9 Balfour Street
Greenwich
NSW 2065
Friday 23rd July 2010

Mr Tom Alegounarias
President, New South Wales Board of Studies
Office of the Board of Studies NSW
GPO Box 5300
Sydney NSW 2001

Dear Mr Alegounarias,

Structure, Content and Coherence in Mathematics and the Senior Curriculum

I am writing to ask the NSW Board of Studies not to accept ACARA's proposals for Years 11–12.

- The NSW Board of Studies should reject the elimination of the 1 unit Extension courses across the curriculum, and should also reject the options of taking 4 units of Mathematics or English in Year 11.
- The NSW Board of Studies should reject the course structure and the content of ACARA's Draft calculus syllabuses for Years 11–12 (Course C: Mathematical Methods, and Course D: Specialist Mathematics).

ACARA's earlier proposals for the K–10 curriculum would cause significant damage to mathematics in NSW, as I detailed in a previous letter. The effects of ACARA's Draft 11–12 documents are considerably more serious, in that across the senior curriculum, they would sweep away structures that have allowed NSW to maintain its traditions of excellence for nearly a century.

We do not yet know what changes ACARA will make to the Drafts of its Years K–10 documents. We do not even know the overall structure of the curriculum that ACARA is proposing for K–10 — we only have Drafts of four isolated subjects — but the intention seems to be to restrict severely the time for other important subjects like languages. We do not yet know what adaptations NSW will make to ACARA's curriculum structure in Years K–10 or to individual syllabuses. We do not even know whether NSW will be allowed to make such adaptations. All this will need considerable consultation. Two issues that need urgent clarification in Years 9–10 for NSW are the retention of Electives, and the retention of the differentiated Mathematics courses.

It is inappropriate for ACARA to be producing Draft Years 11–12 courses, and asking teachers to comment on them, when the serious and complicated issues within subjects and across the curriculum raised by its K–10 Drafts have not been settled. My first recommendation is thus quite straightforward.

> **Recommendation 1 for NSW**
> NSW should insist that all consideration of national Years 11–12 courses be delayed for several years until a national Years K–10 curriculum has been agreed upon, has been adapted by the States, has been established in schools, and has received acceptance by teachers and the general community as a clear improvement on the current situation in all States, including NSW.

In particular, asking teachers to submit responses to Draft senior courses only two months after they have responded to the K–10 Drafts is unreasonable. Most of the teachers that I have spoken to are so busy with current school work that they have not even had to chance to read the senior Drafts.

I will first address the structure of Extension courses in Mathematics and across the curriculum, and then turn to the content details of the calculus courses. As in my May letter on the Draft K–10 Mathematics, I will do my best to recommend constructive solutions for NSW. This will, however, require far more change from ACARA, and far more subsequent adaptation from the NSW Board of Studies, than was the case with the K–10 Drafts.

Please excuse my language of 'units' in many inappropriate places. Units are, of course, peculiar to NSW courses, and are no longer used in course names, but I have found it difficult to write clearly without assigning a notional value of '2 units' to courses in ACARA's proposals and in other States. I will use the term 'university mathematics' to refer to the mainstream university mathematics courses on the path to a Mathematics major, as distinct from one-year service courses.

Part 1: The need for 1 unit courses in the Years 11–12 curriculum

When Professor Room originally designed the NSW calculus courses in the 1960s for the new Wyndham Scheme, he conceived the 3 unit course — Second Level Full — as the basic requirement for entry to university mathematics. A cut-down 2 unit version of this course — Second Level Short — was offered as a basic, but nevertheless quite demanding, calculus course, which could lead to university mathematics provided that a 'bridging course' to Second Level Full was taken before the start of the university year. A 4 unit course — First Level — was offered for exceptionally able and motivated students.

This structure was a brilliant innovation by Room. It recognised that mathematics is the gateway to the sciences, in that progress in most university science subjects requires considerable prior strength in mathematics, and that a course-and-a-half of Mathematics was needed by those proceeding to university mathematics. At the same time, it allowed students to find their own level in mathematics at school. For those who had taken Second Level Short, but whose interests and career choices turned to science after the start of senior school, it allowed re-entry to university mathematics via a bridging course. The effectiveness of this structure has been so convincing that it has been maintained in NSW ever since through all the interminable changes, with the names evolving as follows:

- 'Second Level Short' became '2 Unit Mathematics', then 'Mathematics'.
- 'Second Level Full' became '3 Unit Mathematics', then 'Mathematics + Extension 1'.
- 'First Level' became '4 Unit Mathematics', then 'Mathematics + Extension 1 + Extension 2'.

Year 11 is too early to choose 4 units of Mathematics

One significant change to Room's structure was made in the top course. The original 'First Level' course required 4 units of study in both Year 11 and Year 12, whereas the later '4 unit Mathematics' only began in Year 12. This was a wise move, because students do not know at the end of Year 10 whether they are capable of the extreme demands of 4 units, neither can their Year 10 teachers advise them effectively.

Calculus is exciting, but it demands far more unified and advanced logical thinking, and far sterner algebra, than Year 10 Mathematics. We routinely find that some students thrive on calculus after mediocre performance in Year 10, while others who coped well in in Year 10 cope less well with calculus.

We now have the ideal situation that a capable Year 10 student can confidently be placed in Extension 1 in Year 11. On the basis of their Year 11 result, we can then advise continuing with Extension 1, taking Extension 2, or dropping to 2 unit (or in a few cases to General Mathematics or to no mathematics).

ACARA's Drafts would ask our calculus students to choose Course C (equivalent to 2 units) or Courses C + D (equivalent to 4 units) in Year 11, with no option to take 3 units. This would make things very difficult, because they would not be in a position to make such a choice at the end of Year 10, and we would not have the information to advise them effectively, with the result that many able students who had aimed too low would be stranded in Course C — inadequate for university mathematics — while others who had aimed too high would be forced to drop from Courses C + D to Course C in Year 12. Many of those dropping Course D would, as a result, have serious problems finding enough units for their HSC, exacerbated by the elimination of 1 unit Extension courses in humanities. Every school teacher knows the importance of morale in Year 12, and how badly it can be affected by problems of inappropriate subject choice.

2 units + 4 units is not an option for Mathematics

ACARA has made a vague suggestion that students could take just Course C in Year 11 (2 units), then take Courses C + D in Year 12 (4 units), resulting in a programme equivalent to 3 units for each of two years. This proposal is unworkable for a number of reasons. First, Courses C and D are both extremely poorly structured at the moment, as I will discuss below, but taking half of Course D as suggested would further fragment the mathematics. Secondly, it is too much of a jump from the reasonably straightforward demands of the 2 unit Course C to the intensely demanding Course D. Except for exceptional cases, neither we nor the students could be confident that the right decision was being made. Thirdly, the Victorian experience is that only a handful of students make such a switch — such an option would thus have no appreciable effect on the standards of university mathematics or science in NSW.

Why should NSW be forced to change from an effective system, supported by enthusiastic teachers and students, to the Victorian model with all its problems?

The loss of 3 units of Mathematics would lead to a substantial 'dumbing down' at university

Unfortunately, since my previous letter, ACARA has not modified its intention to exclude 3 units of Mathematics, and I must repeat the remarks I made then. Last year NSW had $5460+3170 = 8630$ school-leavers taking respectively 3 units or 4 units of Mathematics — these are the school-leavers regarded as capable of taking university mathematics. If ACARA's course structure is introduced, with no 3 unit course, the number of school-leavers capable of university mathematics could thus drop to a little over 3000 — a 60% 'dumbing down'. This would ruin the present strength and diversity of the universities in NSW.

According to recent studies of Barrington and Brown (under the auspices of AMSI), NSW schools produce more than 40% of the school-leavers across Australia capable of university mathematics. NSW thus punches well above its weight compared to the other States. Surely ACARA should be using the NSW structures as its model for national Mathematics courses, rather than seeking to tear down what is universally regarded as the most effective model in Australia and replace it with the less effective Victorian model?

Things are worse than the 60% figure quoted above. The present tendencies in Victoria demonstrate that even very able mathematics students are dropping the Specialist Mathematics (4 units) option — the major cause being the forced choice of 4 units in Year 11 — with the result that even Melbourne University can no longer maintain 4 units as its entrance requirement for mathematics, and is having to begin its First Year courses from the 2 unit level. Bridging from 2 units to 4 units over the university vacation is, of course, not a reasonable option. Thus the present enviable standards of mathematics at NSW universities would also inevitably have to drop to accommodate 2 unit HSC students, and with it, the standards of science would drop as well.

Recommendations for the senior calculus courses

These recommendations follow from the discussions above, and are transparently based on Room's model, which has proven so effective for so long.

> **Recommendation 2 for NSW**
>
> The senior calculus courses in NSW should have the following structure. Such courses could be implemented by ACARA, or by the NSW Board of Studies within a National Curriculum structure more flexible than presently envisaged.
>
> - A course of 3 units in each of Years 11 and 12 — a course-and-a-half — should be written first as the standard course for entrance to university mathematics and most of the sciences. This course would develop calculus and related topics systematically. It would be unified, so that virtually any two elements of the course can be related in a problem. It would be as systematic and rigorous as is reasonable for the maturity of the students, with most results proven.
> - A cut-down version of this course should be offered as a 2 unit course in Years 11–12. Basic calculus would be taught systematically, and the course would still form a unity, but rigour would be reduced appropriately in its demands, and the harder topics of the 3 unit course omitted. Bridging courses to 3 unit would be possible over the university vacation.
> - The 3 unit course may for pragmatic reasons be written as the 2 unit course plus a 1 unit Extension, but it should never be conceived or designed as two courses.
> - A demanding 1 unit extension of the 3 unit course should be offered in Year 12 alone, designed in such a way that all four units form a coherent whole. Universities may or may not offer advanced standing to those who have completed it.

The names of the courses proposed by ACARA are dreadful, and being the names of the Victorian courses, seem designed to cause friction and misunderstanding. The three calculus courses I am proposing are justly called 'calculus courses' because the material in them is organised around calculus and its applications. In particular, probability and integration are closely related, so there is no need to add 'Statistics' to the name of the 2 unit course. The present nomenclature in NSW is notoriously confusing, and I would suggest the following straightforward names for them:

- The 2 unit calculus course: 'Basic Calculus'
- The extension to 3 units: 'Advanced Calculus'
- The extension to 4 units: 'Extended Calculus'

Recommendations for the senior non-calculus Mathematics courses

The consensus of those who, unlike me, are experienced in the present NSW General Mathematics course seems to be that ACARA's Course B is too difficult for the candidature.

Unfortunately NSW abolished its remedial Mathematics course in the New HSC. With the present policy of extending the school leaving age, a remedial Mathematics course is urgently required.

It is typical of ACARA's approach that there is no whole-curriculum context for these non-calculus courses. At best, one would expect a variety of trade courses in which the particular mathematics appropriate for each trade was taught. It is impossible for anyone to evaluate the suitability of these courses, particularly of Course A, without knowing what the Years 11–12 curriculum is to be for students with only basic ability in mathematics.

ACARA proposes that students be allowed to take Courses B and C in Mathematics together in Years 11–12. The world has so many interesting and useful things to study that taking two courses at ordinary level in the one discipline should be entirely out of the question in a school curriculum. This extreme form of the abolition of the breadth requirement cannot be in any student's educational interest. One would have expected Victorian voices to have spoken strongly against this practice in their State. Instead they seem to be championing it for the rest of Australia.

> **Recommendation 3 for NSW**
> The senior non-calculus Mathematics courses should have the following structure:
>
> - A remedial Mathematics course flexible enough to be tailored to the abilities and needs of individual classes and students, and particularly to vocational subjects.
> - A general purpose, and reasonably straightforward, non-calculus course to review and maintain the methods of Years 7–10 Mathematics, and to extend them in areas useful to non-mathematical tertiary students and to the demands of business and modern life. There should again be flexibility so that the course can be tailored to various abilities and needs.
> - There should be no option to take simultaneously a calculus and a non-calculus Mathematics course.

Year 11 is too early to choose 4 units of any subject
ACARA proposes the same unwieldy choice of 2 units or 4 units of English in Year 11. Every time a student chose 4 units of either subject in Year 11 they would drop some other subject, and if they then had to drop back to 2 units, they would have subject-choice problems in Year 12.

Such restriction of subjects in Year 11 would be a huge blow to languages and humanities, which would suffer the loss of their most able students. But for the students themselves, it is too soon for such specialisation in Year 11, where 3 units even of key subjects is quite enough. Children's interests and talents are changing so quickly at this age that in Year 10 they are not in a position to restrict their HSC choices to this extent. What we have now in NSW works very well — partial subject choice in Year 11, then final subject choice after a proper encounter with Year 11 work.

It was already a disappointment that the New HSC in NSW lost much of its generalist nature when it dropped the breadth requirement, which required students seeking matriculation to take English, a unit of Mathematics/Science, and a unit a of another humanity. Allowing 4 units of Mathematics or English in Year 11 would only make a bad situation much worse.

ACARA seems to have no place for the present Extension 2 English course, in which students write a major work. This has been a successful innovation in schools, and if NSW cannot offer this course in future, it would be great loss for the many talented students who have taken the opportunity for creative writing that it has offered. The situation is another indication of the rigidity and lack of imagination in ACARA's thinking.

> **Recommendation 4 for NSW**
>
> - There should also be no option to take 4 units of English in Year 11.
> - Some form of the present Extension 2 English course, in which students write a major work, should be retained as a 1 unit course in Year 12 alone.

The loss of 1 unit Extension courses in the humanities
It would seem from ACARA's unfortunately incomplete proposals that all the other NSW Year 12 Extension courses in languages and the humanities would disappear. These courses, as with those in English and Mathematics, had their origins in the old Leaving Certificate Honours courses, which evolved into Level 1 courses, then into 3 unit courses, then into their present form. They

have been the backbone of high achievement in NSW schools since before World War II. They also add flexibility to the choice of subjects for the HSC, allowing slightly more specialisation in the final year, and perhaps the dropping of a subject that has failed to inspire or proven disappointing in results.

Students with a Language Extension often enter the second year of the language at university, and more generally, the enthusiasm that these Extension courses routinely generate amongst the able students who take them means that they embark on university study with an impressive intellectual enthusiasm. They also give able teachers the opportunity to present their discipline at a higher level — this is a rewarding experience for them personally, and within a school common room helps to preserve the enthusiasm of teachers for their subject, to the benefit of all their classes. A single unit in a single year can have a great effect on a school.

The fact that other States do not have such courses is no reason at all to eliminate such a highly successful feature from the NSW curriculum. Again, I am amazed that ACARA has not adopted a structure that has proven effective and rewarding to teachers and students for so many decades. They seem to be blind to the strengths of the system that they have set out to improve.

> **Recommendation 5 for NSW**
>
> - Extension courses of 1 unit — half a course — should be offered in Year 12 alone in selected humanities, including Modern and Classical Languages, History, Music and Art. (Unfortunately NSW currently has no Extension course in Art.)
> - Such courses could be implemented by ACARA, or by the NSW Board of Studies.

Languages and Electives in Years 7–10

I have included at the end of this letter a page that was distributed freely at a recent meeting of Geography teachers. It is only a draft, and the rest of the document is unknown. It gives hours per week in subjects, but the hours add to less than the total hours available, leaving some freedom of movement. The document raises all sorts of problems, as I am sure you are aware, but in the context of the present discussion, the implications for Languages and Electives seem quite serious.

In each of Years 7–10, only 75 hours per year are assigned to Languages. This is a silly figure, because no one can learn a language in less than 2 hours per week. The HSC course numbers in NSW indicate that we are not doing at all well with foreign languages at the moment, and that attention needs to be given to the situation.

The draft page also seems to leave no room for the successful NSW structure of three Elective subjects in Years 9–10.

I would have thought that the construction of a Years 7–10 curriculum would begin by looking at the whole child, and trying to create an overview of what things children of different abilities and interests should be learning at school during these years. One of the first questions to ask would be, ‘How do we encourage children in Years 7–10 to take foreign languages and persist with at least one language until they have a useful knowledge of it?’

> **Recommendation 6 for NSW**
>
> - NSW should be given the opportunity to implement a system of Electives in Years 9–10 that is at least as effective as the present NSW system.
> - Whether or not any National Curriculum is implemented, NSW should review how languages can be encouraged in Years 7–10 and in senior school.

The draft page seems to confirm criticisms of ACARA’s processes — there is no coherent view of the curriculum, and no coherent view of the child.

Part 2: Reconstructing Course C coherently

Professor Room's structure of the 2 unit and 3 unit courses still provides us today with the key for constructing courses to prepare students for university mathematics. Mathematics is very demanding in terms of ability and the time required to master it. It is not possible in a 2 unit course to provide a sufficient basis for university mathematics. A 3 unit course is required as the basic academic course in mathematics, and should be written first.

This 3 unit course should form a coherent unity, based on elementary calculus, which is universally required in science, economics and statistics, and yet unlike most other branches of mathematics can be taught satisfactorily to reasonably able students at school. Everything studied previously — arithmetic, geometry, mensuration, trigonometry, algebra, coordinate geometry, graphs — would be reviewed, extended, and drawn upon to support the structure of calculus. Later in the course, contrasting applications of calculus would be presented, particularly in motion, rates of change and probability — and statistics, if one is courageous.

After this has been done, the course can be cut down to a 2 unit course by omitting the more difficult topics and topic items, and lessening the demands of the proofs and the difficulties of the problems. The resulting course, however, would still be totally coherent, and still based on calculus, which also admits effective study at this less demanding level. Despite falling short of providing entry to university mathematics, such a 2 unit course serves three significant purposes:

- It provides late developers — late in terms of ability, interest or career decisions — to re-enter university mathematics via a bridging course.
- Its basic calculus provides students proceeding to less demanding sciences or to qualitative economics with sufficient mathematical knowledge and skills for their needs.
- It allows students who will never again use advanced mathematics to follow their interests in mathematics at school, and so come to understand something of calculus, which is one of the great intellectual creations of the modern world.

ACARA has failed to follow this course structure, and as a result, its Draft Course C is a failure from the point of view of NSW mathematics teachers and academics. It has three fatal flaws:

- It is too hard for the present 2 unit candidature in NSW, yet it is not advanced enough to form a basis for university mathematics as understood by NSW universities.
- Its content is incoherent. This is a result of trying to cover too much ground in too little time, but is also the result of trying to cut and paste together the contradictory approaches of the present NSW and Victorian syllabuses.
- The Draft shows no conception of the students — either of the way in which their abilities and understandings are to be developed, or of their imaginative involvement in the ideas. It gives the impression of a mass of mathematical items written down without any real thought about the student and teacher who are going to be so deeply involved in it for two years. Again, this seems to be caused by endless cutting and pasting.

The result is a course that looks as dull as dishwater. It achieves no sensible goal, it has no coherent structure, and it fails to bring any life and colour to calculus.

I would recommend strongly that ACARA, and failing them NSW, should abandon this futile attempt to combine the goals of the present 2 unit and 3 unit courses in NSW into a single course. This approach will not work, because it does not fit the structure of the mathematics, and it does not fit the needs and abilities of the students.

I will now discuss topics in turn, explaining the failings of the Draft Course C, and explaining how the topic should be constructed within a 2 unit, and within a 3 unit course. I cannot salvage from the Draft a single course that both fits the candidature and prepares students for university study.

To give the flavour of the difference in demand between a 2 unit and a 3 unit course, here are a few of the topics and topic-items that, apart from any considerations of restricted time, are too hard for 2 unit candidates, but should be studied by those heading for university mathematics:

- Inequalities, and absolute value (apart, perhaps, from basic material)
- Circle geometry, and intercepts theorems in geometry
- Mathematical induction
- First principles differentiation (although 2 unit students should see it)
- Limit proof of the fundamental theorem of calculus (2 unit students should see it)
- Limit proof that $\dfrac{d}{dx}\, e^x = e^x$, or that $\displaystyle\int \frac{1}{x}\, dx = \log_e x$ (2 unit students should see it)
- Differentiation and integration of exponential functions with bases other than e
- Limit proof that the derivative of $\sin x$ is $\cos x$ (the graphical explanation will do at 2 unit)
- The inverse trigonometric functions, and the calculus of $\sec x$, $\operatorname{cosec} x$ and $\cot x$
- Integration by substitution, including systematic use of the product rule in reverse
- Superannuation and housing loans as applications of GPs
- The form $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$ for the second derivative in motion
- Identities on the coefficients of the binomial expansion (apart from the basic identities)
- Statistical inference

Review and extension of Year 10 material in the new context of graphs

Polynomials, binomial expansions, the binomial theorem and ${}^n\mathrm{C}_r$ are well known as hard topics in the NSW Extension 1 course, yet they are presented in the opening Topic 1 of Unit 1 of the Draft Course C. Starting a new 2 unit class in Year 11 with this material would be a disaster. It is hard to have confidence in ACARA as a designer of syllabuses when a course intended for the present NSW 2 unit candidature begins like this. The much easier, but still tricky, idea of logarithms is also misplaced in the opening lessons of the opening topic with a new class.

Perhaps the writers intend teachers to reorganise material — it is hard to comment when so much is unknown about intentions — but the first few topics of the Draft have further serious structural problems that are indicative of quite different, and much deeper, problems with the Draft.

- The five ideas of indices, logarithms, surds, polynomials, and binomial powers are presented with no graphical background. This is inappropriate in a calculus course, where graphs should be used as the organising principle of the material, and everything possible should be clearly visible in the graphs.
- There is no recapitulation and extension of a great deal of difficult Year 10 material. I have never met a Year 11 class, at any level of ability, who would not be demoralised if I launched immediately into new material without some such review and extension.

The first task in any Year 11 calculus course is to review and extend Year 10 material in the context of graphs. The order below is not intended to be definitive, but in NSW terms would be suitable for a 'struggling Extension 1' class. Classes, students, and the insights of teachers differ so much that any prescriptive order of topics should be avoided throughout the courses as far as possible.

1. Students may or may not need a review of factoring (HCF, difference of squares, quadratics, grouping), completing the square, surds, and real, rational and irrational numbers.

2. The concepts of functions and relations, function notation, the graph of a function or relation, and domain and range.
3. A review of known graphs, including at least:
   - Linear functions and relations (but not yet coordinate geometry)
   - Quadratic functions (but only straightforward examples for now)
   - Higher powers of x, and $y = \sqrt{x}$
   - Circles and semicircles
   - $xy = a$, and associated asymptotes
   - $y = a^x$, and the associated asymptote
4. Introduction to inverse functions and their graphs and algebra.
5. Transformations of known graphs by translation and by reflection in either axis, and in 3 unit by stretching.
6. Further work on graphs, including testing the sign of a function, odd and even functions, asymptotes, regions, using graphs to solve inequations, and at 3 unit level, the absolute value function and inequalities. Here the methods of factoring are being used to draw the graphs.
7. Recapitulation of trigonometry in degrees, problems in right-angled trigonometry and the sine and cosine rule and area formula, trigonometric identities, and the extension of the trigonometric functions to any angle, all still in degrees. Now the graphs can be drawn and some of their symmetries discussed, so that the object of study changes from triangles to waves, and trigonometry is brought into the graphing structures built up so far.
8. A review of the geometry of congruence, similarity and special quadrilaterals. Intercepts and the geometry of circles, including the converse circle theorems, are suitable for 3 unit, but circles should be delayed until much later because their study now would disturb the flow of the early part of the course.
9. A review of coordinate geometry, including point–gradient form and simultaneous equations. Problems in the coordinate plane should review the area formula from Years 7–10 mensuration, and also involve trigonometry and the sine and cosine rules and area formula, particularly bringing out the relationship $\tan\theta$ = gradient in preparation for calculus. The coordinate plane is the playground where graphs and calculus take place, and fluency here is essential.

   Perpendicular distance, and the use of coordinate geometry to prove geometric theorems, are suitable for 3 unit, as is ratio division of a line, which relates in interesting ways to intercepts in geometry. (We would all be relieved, however, if the current NSW item ‘lines through the intersection of two given lines’ were omitted.)
10. Now the way is clear to go back and review indices, and review or introduce logarithms — I recommended in my previous letter that introductory logarithms should be in an Advanced Year 10 course. This time, however, the discussion should take place in the context of graphs of exponential and logarithmic functions, and of graphs of inverse functions, so that the concepts make geometric sense as well as algebraic and arithmetic sense.
11. More review and extension of quadratic functions is needed. Current NSW practice is to delay this a little so that calculus can begin, and this has proven to be an effective approach. Some of the quadratic items are: the relation between the factored form and the graph, and between the competed square form and the graph, maximisation and minimisation, and the discriminant. The sum and product of roots, the subsequent parametric study of the parabola, and the more general idea of locus, are suitable for 3 unit.

The following criticisms of the Draft Course C are relevant in this context of how the course should begin, and of the necessary review and extension of Year 10 material in the new context of graphs.

- There is no review of coordinate geometry. One cannot do calculus without a firm grasp of the coordinate plane and problems associated with the graphs of straight lines.
- There is no review of elementary Euclidean geometry. These concepts of congruence and similarity define what the coordinate plane is, and the basic shapes of special triangles and quadrilaterals recur throughout the course in the theory and in the problems.
- There is no review of factoring. Apart from very able students, few enter Year 11with a solid grasp of factoring. They won't be able to apply calculus if they can't factor.
- There is no review of problems involving the sine and cosine rules and the area formula, which are rarely fully assimilated in Year 10. Moreover, the abstract and confusing notion of radians should not be introduced at this stage (Unit 2 Topic 1), but should wait until the calculus of trigonometric functions, when radians makes things easier because then $y = \sin x$ has gradient 1 at the origin and $\dfrac{d}{dx}(\sin x) = \cos x$. Before that, radians are complicated and unmotivated.
- There is no work on trigonometric identities, which are vital in the calculus of the trigonometric functions. Even the identities based on Pythagoras' theorem — the most important single theorem in the whole of school mathematics — seem to be missing.
- Differentiation comes too early (U1, T3). A student needs much more experience of graphs before it becomes clear that there is a problem with finding the gradient of a graph at different points on it, which is the motivation for the derivative. Yet the derivative precedes trigonometry and its graphs (U2, T1), graphs of exponential and logarithmic functions, and simple properties and transformations of graphs (U2, T2).

Matrices

A matrix is a movement in space. This concept, however, and the accompanying linear algebra, are too subtle for school mathematics (which I will mention again in the context of Course D). The matrices in Course C are thus purely place-holders of the coefficients and constants in a block of simultaneous equations.

Solving three simultaneous equations is reasonably straightforward when the equations are written in the usual manner in terms of x, y and z. There is no point in asking students to carry out this simple procedure when the block of coefficients and constants is divorced from the variables. No appreciable computation time is saved, and no insight is gained into the algebra of linear transformations, or inverse matrices, or anything that is useful at school.

A great deal, however, is lost. First, few of the present 2 unit and Extension 1 students will ever have the faintest idea of what is going on, because as soon as the variables are removed, the straightforward process of eliminating each variable in turn becomes impenetrably abstract for them. Secondly, on the few occasions elsewhere in the courses when three simultaneous equations need to be solved, few students will remember what for them is such an obscure process, and will sensibly revert to the variables. Thirdly, it will take much wasted time and effort to teach, and thus irritate teachers and students.

Even using variables, it would be unwise to deal with contradictory or redundant systems of simultaneous equations, because without a proper treatment of three-dimensional coordinate geometry, there will be no geometric insight into what is going on when the equations describe objects such as triangular prisms, or three planes meeting in a line. The task will thus be dull and mechanical.

This topic is a most ill-advised proposal. Vectors are a reasonable topic to teach at school, because their geometry is clear and imaginative, and they are useful in other parts of the course, but the geometry of matrices belongs in university mathematics.

Recommendation 7 for NSW

- The 2 unit and 3 unit courses should begin with a review and extension of Year 10 arithmetic, algebra, geometry, mensuration, trigonometry, coordinate geometry and graphs, all presented in the new context of graphs, as detailed above.
- Some more difficult items are only suitable for 3 unit, as detailed above.
- Systems of two, three, or in simpler cases four, simultaneous equations should be taught in this context, but without any reference to matrices.

Sequences and series

The Draft Course C omits sequences and series, which are an essential part of calculus. Presumably this is because of the rush to fit too many things into the course. The topic thus provides an excellent example of why the 3 unit course is needed. Sequences and series should be in both the 2 unit and 3 unit courses for various reasons.

- Most significantly, both courses will study exponential functions based on $y = e^x$. Having e as the base of an exponential functions is exceedingly abstract, and students will have no chance of understanding this idea unless they have earlier dealt with exponential functions with friendly bases like 2 and corresponding practical problems. In situations when scientists are dealing with exponential functions, their measurements tend to be made with base 2, as for example with the half-lives of radioactive decay. We pass to base e in calculus because the situation is simpler when the gradient at the y-intercept is exactly 1, so that e^x is its own derivative, a situation precisely analogous to radian measure in trigonometry. It is most important, therefore, that the student previously works with GPs to become familiar with the behaviour of exponential functions before passing to exponential functions base e as required in calculus.
- APs and GPs are the discrete versions of linear and exponential functions. Most discrete mathematics is too hard for school, but APs and GPs are quite straightforward and extremely useful. It is not appropriate to be studying more complicated discrete objects such as the binomial theorem and ${}^n\mathrm{C}_r$ and the Pascal triangle before mastering APs and GPs.
- Simple and fundamental ideas in arithmetic and algebra, such as the limiting sum of a recurring decimal, and the factorisation of the difference of nth powers, rely on the sum or the limiting sum of a GP. The factoring of the difference of nth powers, in particular, completes the standard methods of factoring begun in the opening review, and provides by far the most straightforward way to establish the derivative of x^n.

  The Draft Course C envisages establishing the derivative of x^n using the binomial theorem. This is a far more complicated approach, however, because proving the binomial theorem is difficult, which is one reason why the start of the Draft Course C is so misjudged.
- It is unwise to deal with the tangent as the limit of a secant before dealing with the more approachable limiting sum of a convergent GP.
- Arithmetic and geometric means, which arise routinely in statistics and in geometry, rely on APs and GPs.
- GPs are necessary for calculations with superannuation and housing loans, which should be part of everyone's general education. These are, however, too difficult for the 2 unit course.
- Mathematical induction belongs naturally within a sequences and series topic, pragmatically because it is usually introduced as a method of proving the sums of some slightly more complicated series, and theoretically because mathematical induction is a recursive idea and strictly speaking is needed to justify the logic of the formulae for APs and GPs.

Mathematical induction is necessary preparation for university mathematics, yet it is too hard for 2 unit. Once again, the 3 unit course is required as the proper place for the topic.

> **Recommendation 8 for NSW**
> - APs and GPs, including limiting sums and arithmetic and geometric means, should form part of the 2 unit course.
> - Practical applications of both sequences should be given, except that superannuation and housing loans should be restricted to the 3 unit course.
> - Mathematical induction belongs in 3 unit in association with this topic.

Calculus in the Draft Course C

With calculus, we come to the heart of the course, and to the heart of the disorganisation of the Draft Course C. The problems in the presentation of calculus come under four headings.

1. The imaginative centre of differentiation is its relationship with curve-sketching, yet this relationship is not presented in a coherent way.
2. Integration is introduced in a seriously confused unit (U3, T1), which mixes up integration with further differentiation, and with the logarithmic and exponential functions.
3. The exposition of the special functions — exponential, logarithmic and trigonometric — is confused because it is combined with the exposition of differentiation and integration.
4. The attempt is made to develop motion in each part of calculus, whereas students find this application very challenging and need to study it as an application of calculus in a separate topic.

Differentiation

Turning first to differentiation. there are two stages in the teaching process, corresponding hopefully to two topics separated by other material — in NSW at present this intervening material is usually quadratic functions, but there is no canonical order here. All this should be done purely with functions built up from powers of x — there are too many conceptual problems associated with introducing e^x, $\log_e x$ or $\sin x$ too early.

STAGE 1: INTRODUCING THE DERIVATIVE AND ITS COMPUTATION:

The derivative is introduced as the gradient of the tangent at a point. The required limiting process needs explanation at 2 unit, but it probably only makes sense to examine the limit at 3 unit. Much more important than dwelling on the limit is the drawing and computation of tangents at points on the curve, and the identification of points where the tangent has a given gradient. Such exercises make concrete the idea of the derivative as the gradient of the tangent.

There is no reason to avoid the chain, product and quotient rules at this stage. They make a good pause in the forward movement of the course at a critical stage, they continue to review important algebraic skills, and they add to the stock of functions that can be examined in terms of tangents.

If the formula $\delta y \doteqdot \dfrac{dy}{dx}\,\delta x$ is to be included in the course, it belongs here, as part of the explanation of the derivative, to be used throughout calculus, not introduced in the very last topic of the course.

STAGE 2: APPLYING DIFFERENTIATION TO CURVE-SKETCHING AND MAXIMISATION:

After an interlude, the derivative can then be let loose on a large range of functions that could not be satisfactorily sketched using the ‘known functions and their transformations’ methods of the opening review section. The first candidates are polynomials, where the turning points, and then the inflexions, of selected cubics and quartics yield to factoring techniques reviewed earlier.

Progressively more complicated functions involving fractional indices, reciprocals, products, quotients, and compound functions can then be covered, according to the class's abilities. Obviously 3 unit can progress much further with such things.

It was a disappointment to see points of inflexion left so late (U3, T1). The surprise that one can find inflexions is a wonderful way to convince students of the cleverness of calculus.

The other great application of calculus at this stage is finding maxima and minima, and the solution of a vast range of practical problems using calculus, along the lines of 'Farmer Brown has wire for 200 metres of fencing and wants to maximise the area of a rectangular paddock built against an existing wall.' These problems are excellent exercises in setting up functions, as well as in maximisation. (Problems giving rise to quadratic functions have, of course, alternative algebraic approaches using completing the square or locating the vertex midway between the roots.)

It was a great surprise to see that these problems have been omitted from the Draft (or perhaps this is what the words 'Optimisation over finite intervals' refer to in U4, T3 — if so, the item comes too late). They convince everyone of the practicality of calculus, and they are a canonical part of any introductory calculus course.

Recommendation 9 for NSW

- The derivative should be introduced in two separated sections, before any mention of the integral or of special functions. The first section introduces the derivative, deals with tangents and their gradients, and drills the chain, product and quotient rules. The second section applies the derivative to curve-sketching and to maximisation problems.
- First-principles differentiation should only be fully developed in the 3 unit course, and the range of functions to be sketched should be restricted in the 2 unit course.

Integration

Integration, and the notation for the integral, is based on the problem of finding areas under a curve. The brilliant idea that it is the reverse process of differentiation means, however, that it is better to introduce anti-differentiation first, and some of its algebraic methods, before turning to areas. This means problems of the form, 'If $f'(x) = x^2$ and $f(2) = 0$, find $f(x)$'. These problems and their methods lead directly into later practical problems in motion and rates of change.

Integration needs at least some support from complementary graphical work. Simpson's rule is too technical for school, but the trapezoidal rule, on one interval at a time only, is excellent, because the second derivative can be used to find whether the approximation overshoots or undershoots the true value.

Volumes also need to be covered, for their intrinsic interest, and for the logic of the school mathematics courses. It is only using volumes of integration that students can obtain proper proofs of the foundation formulae for the volumes of spheres and cones.

Unfortunately the Draft muddles up integration with later work on differentiation and with the beginnings of the exponential and logarithmic functions. It should be a topic on its own. The fundamental theorem of calculus should be proven in both courses, but only in the 3 unit course is it appropriate to examine it in the form $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. This formula is, of course, vital in any rigorous approach to the normal distribution.

This is also the time to introduce the simple linear extensions $\int (ax+b)^n \, dx$ into 2 unit, and the more general form $\int y^n \frac{dy}{dx} \, dx$ into 3 unit (in whatever notation).

> **Recommendation 10 for NSW**
> - The integral should proceed as a topic on its own, first through the anti-derivative, then through areas and volumes, supported by the trapezoidal rule.
> - The form $\int (ax+b)^n \, dx$ belongs in 2 unit, and $\int y^n \frac{dy}{dx} \, dx$ in 3 unit.
> - The fundamental theorem of calculus should be presented in 2 unit and used to develop the definite integral. It can be developed further in 3 unit.

The exponential, logarithmic and trigonometric functions

The exponential and logarithmic functions are some of the hardest material in the present 2 unit course. Effective teaching of them requires that besides an earlier thorough treatment on logarithms and indices and their graphs, and another session on trigonometric functions and their graphs, students have already assimilated the derivative and the integral, their graphical interpretations, associated algebraic skills, and curve-sketching and maximisation. Then each aspect of calculus can be expounded as it applies to these two special functions.

A separate topic should deal exclusively with the demanding calculus of the trigonometric functions. This is now the place for radians, as I remarked earlier, and once again, each already assimilated aspect of calculus can be applied to the trigonometric functions.

The approach of the Draft to these special functions — mixing up their exposition with further exposition of the derivative and the integral — will not work in the classroom.

Where is the clear exposition of exponential and logarithmic functions? One would expect an ordered sequence along the lines of:

- The derivative of exponential functions
- Applications of the derivative of exponential functions
- Integration with exponential functions
- Applications of integration
- The derivative of logarithmic functions
- Applications of the derivative of logarithmic functions
- Integration with reciprocal functions
- Applications of integration

The dotpoints given in the Draft give no indication of what depth of treatment is required.

The remarks applicable to the calculus of the trigonometric functions are analogous in every way.

A small point concerns the primitives $\int f(ax+b) \, dx$, where $f(x)$ is a function whose primitive is known. These are only introduced in the very last Topic, so are we to conclude that $\int e^{ax+b} \, dx$ should not be introduced until this point, yet $\int \frac{dx}{ax+b}$ is introduced with the logarithmic function? The Draft has many ambiguities and misunderstandings like this.

Recommendation 11 for NSW

- The exponential and logarithmic functions, and then the trigonometric functions, should each be expounded systematically within a section of their own.
- The following item are only suitable for 3 unit:
  - The limit proof that $\dfrac{d}{dx}\,e^x = e^x$, or alternatively that $\displaystyle\int \frac{1}{x}\,dx = \log_e x$
  - The limit proof that the derivative of $\sin x$ is $\cos x$
    (Graphical proofs of these two are all that most 2 unit students can cope with)
  - The calculus of exponential functions with bases other than e
  - Trigonometry with compound angles (apart, perhaps, from $\sin 2\theta$ and $\cos 2\theta$)
  - The calculus of $\sec x$, $\operatorname{cosec} x$ and $\tan x$ (apart from the derivative of $\tan\theta$)
  - The calculus of the inverse trig functions $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$
  - The more general study of inverse functions of restricted functions

Applications of calculus

The two immediate applications of calculus are motion, and rates of change. Both are difficult for students to understand and need great care in their exposition. The presentation of each is confused in the Draft.

MOTION:

Students find the application of calculus to motion far more difficult than one may expect. Even the basic idea that a stone thrown vertically has zero velocity, but non-zero acceleration, at the top of its flight, does not come easily to 2 unit students. The Draft's approach of distributing motion around the calculus syllabus as an example to help students understand calculus will have the opposite effect, because motion presented in little bits will only confuse them. One has to develop calculus first, then spend a sustained period of teaching time to develop motion as a model of calculus.

It is a wonderful model, because the first derivative is velocity, which we can see, and the second derivative is acceleration, which we can feel. No other simple model of calculus leads to such a clear representation of the second derivative. The indefinite integral is modelled whenever the acceleration is known, which relates the topic easily to everyday experience of forces.

RATES OF CHANGE:

Similar remarks apply to rates of change. The current NSW syllabuses also handle this topic badly, distributing it across several places, but never clearly presenting what should be the initial idea, 'Given Q as a function of t, differentiate to find the rate $\dfrac{dQ}{dt}$ as a function of t.' As with motion, there should be a connected topic on rates, starting with the derivative as a tool to calculate rates, then using the indefinite integral to reverse the process. Finally, the extraordinary case where $\dfrac{dQ}{dt}$ is proportional to Q illustrates why it was so important to introduce the special function $y = e^x$ (analogous to the use of $\sin x$ and $\cos x$ to solve $\ddot{x} = -x$ in 3 unit).

Related rates using the chain rule, listed in the last topic of the Draft, has proven to be a demanding 3 unit topic in NSW. It is unsuitable for 2 unit.

One important benefit of expounding each of these topics late is that all the methods of calculus, including the special functions, are brought together and applied within the one topic. They each provide a marvellous review of most of the course.

Recommendation 12 for NSW

- The applications of calculus to motion, and to rates of change, should each be expounded separately and systematically after calculus has been developed.
- In each topic, some items are only suitable for 3 unit:
  - Related rates of change using the chain rule is unsuitable for 2 unit.
  - In 2 unit, one would normally write down the general solution of $\dfrac{dQ}{dt} = kQ$ and ask students to prove that it is a solution.
  - The form $\ddot{x} = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$ is unsuitable for 2 unit. Thus simple harmonic motion at 2 unit level can only deal with the time equations.
  - NSW currently introduces vectors into the 3 unit course by teaching projectile motion. This provides an excellent practical introduction to vectors.

Polynomials and the binomial theorem

The current NSW Extension 1 course deals at length with the factoring of obscure polynomials by the factor theorem, yet it and the Draft inexplicably omit the factoring of the difference $x^n - a^n$ of nth powers, which is a vital part of so much other mathematics.

At 2 unit, the difference of nth powers is all that is required, and polynomials should otherwise be restricted to sketching factored polynomials, with the help of calculus when prompted. The remainder and factor theorem belong only in 3 unit — but not in the opening topic! The 3 unit course should also contain the extension of the quadratic theory of sum and product of roots to cubics and quartics, which is missing from the Draft.

Counting with ${}^n\mathrm{C}_r$ and the associated binomial theorem would be very difficult for present NSW 2 unit students, but the arguments for including them are unassailable. The treatment in the Draft, however, besides being misplaced, is muddled in its approach, and seems to have no conception of the difficulty of the material even for present Extension 1 students. Combining counting and polynomials in successive dotpoints will not work in the classroom.

The counting approach to ${}^n\mathrm{C}_r$ seems best — most of us have struggled and failed with the abstract NSW approach through the coefficients of $(1+x)^n$. Thus what is needed, for the sake of probability theory as well as for ${}^n\mathrm{C}_r$, is a concise, self-contained topic on counting as it applies to probability, with the proof and application of the formula ${}^n\mathrm{C}_r = \dfrac{n!}{r!\,(n-r)!}$ as its centre-piece, followed by the application of counting to probability problems, and concluding with binomial probability. This will prepare the ground for the binomial distribution. The material would need to be quite restricted at 2 unit.

A separate topic on $(x+y)^n$ should follow this (not precede it as presently in NSW), accompanied by the straightforward counting explanation of why the coefficients in the expansion are ${}^n\mathrm{C}_r$. This will avoid the opaque proof of 'the binomial theorem' in the current NSW Extension 1 course. Such material is extremely difficult at 2 unit, and should be restricted to the basics — the recursion formula for the Pascal triangle, and the formulae for the sums and the alternating sums of its rows.

Further identities on the coefficients are only appropriate at 3 unit. The Draft, as usual, is vague at this point, and omits identities such as $\displaystyle\sum_{r=0}^{n} r \times {}^n\mathrm{C}_r = n \times 2^{n-1}$, obtained by differentiation. Such identities are closely related to the mean of the binomial distribution, which should be in a separate topic. The three approaches to identities required are: substitution, differentiation and integration, and writing the original binomial expansion as a product.

> **Recommendation 13 for NSW**
> - The 3 unit course should have a polynomial topic that extends to the remainder and factor theorem and the sums of products of roots. At 2 unit, all that is required is the difference of nth powers and sketching factored polynomials, both of which will have been covered earlier.
> - The 3 unit course should have a topic before the binomial distribution on counting and probability, centred on the formula for ${}^n\mathrm{C}_r$ and concluding with binomial probability. This material is necessary for 2 unit, but will need to be quite restricted.
> - The 3 unit course should have a topic on the binomial expansion that concludes with identities on the coefficients, as detailed above. The 2 unit version, much restricted, can probably go into the previous topic.

Statistics — Leave statistical inference to the 3 unit course

I believe that it is just possible for the binomial and the normal distribution to be presented to current 2 unit students in NSW, provided that the treatment is kept to the basics, that it is presented systematically with as few leaps in logic as possible, and that the material is unified as far as possible with the rest of the course.

Such a change would require considerable re-education of 2 unit teachers, many of whom are insecure with this level of mathematics. Many will presently have no understanding of 'random variables', or 'probability density functions' or the representation of probability as an integral. Many will not have met ${}^n\mathrm{C}_r$, or Bernoulli trials, or variance. Without a great deal of respectful and carefully structured training in these ideas, the new topics will not succeed.

Statistical inference, however, is beyond the present 2 unit candidature. Distinguishing between the sample mean and the true mean, regarding the sampling mean as a new random variable, and confidence intervals, are completely out of their reach, even if it is only done in the simplest distributions, as in the Draft.

Statistical inference is also beyond the State's 2 unit teachers. Whereas a well-structured and well-resourced teacher-training programme could lift our present 2 unit teachers to be able to teach the binomial and normal distributions, adding statistical inference would make such a task impossible.

Once again Room's structure gives the solution — leave statistical inference to the 3 unit course. The arguments for including statistical inference in the 3 unit course are reasonably sound:
- Students preparing for university mathematics need preparation in proper statistics, not just in probability.
- These students are more able, and better motivated, to cope with a really difficult new idea.
- There are enough mathematically able teachers of Extension 1 in NSW for a well-structured and well-resourced inservice programme to succeed.

I cannot be certain that statistical inference will succeed even in the 3 unit course, but the Draft's proposals here, in contrast to other sections, are severely restricted, thoughtfully presented and seem sensible in terms of the classroom, and I do think that an experiment along these lines is worth risking. I have only a few alterations to suggest.

THE BINOMIAL DISTRIBUTION:
I have already commented on the impossibility of 2 unit having all but the most basic identities on the binomial coefficients. The formulae for the mean and variance of the binomial may therefore not be able to be proven as rigorously in 2 unit as in 3 unit, the other paths to a proof being almost as difficult.

Could the words 'and' and 'or' be used for events? School students have far more difficulty with logic than academics seem to believe is possible, and the notations

$$P(\text{tall and blue-eyed}) \qquad \text{and} \qquad (\text{tall or blue-eyed})$$

will give students a great deal more intuitive sense of what is going on than the abstract forms

$$P(\text{tall} \cap \text{blue-eyed}) \qquad \text{and} \qquad (\text{tall} \cup \text{blue-eyed})$$

which they will constantly misuse and misunderstand.

Defining 'independence' by $P(AB) = P(A) \times P(B)$ may be wonderful in axiomatic probability theory, but it is meaningless to school-children. Can we please stick to probability as an intuitively defined idea at school, and define two stages of an experiment to be independent if the outcome of the first stage does not affect the outcome of the second? We can then deal with sampling without replacement as a separate case.

THE NORMAL DISTRIBUTION:
There is no reason for the Draft to avoid the formula for the normal distribution. One of the main concerns of any senior syllabus writer should be to unify the syllabus and tie the various concepts together. As a standard curve-sketching exercise, 2 unit students can be asked to sketch $y = e^{-\frac{1}{2}x^2}$. The second derivative is available, and the inflexions are at $x = 1$ and $x = -1$, precisely one standard deviation from the mean in the normal distribution, which makes things more geometrically intuitive. We cannot prove that the limit of the area is $\sqrt{2\pi}$ except with the very top students who can follow a long sandwiching argument in three dimensions. This result will unfortunately need to be given as a fact, but the convergence of the area can easily be obtained by comparison with $\int_1^N e^{-\frac{1}{2}x}\,dx$, which can be quickly computed. More generally, I have never seen any good reason why simple improper integrals should not be part of our courses.

The stretching and shifting of the standard normal distribution for other values of the mean and standard deviation can then be related to the transformations of curves, which by this time should be a well-rehearsed procedure.

> **Recommendation 14 for NSW**
> - The binomial and normal distribution can be 2 unit content, but statistical inference is only suitable for 3 unit.
> - The material should be better integrated into the rest of the material in the course using the formula for the normal distribution, as detailed above.
> - Some definitions and notations in discrete probability should be adjusted, as detailed above.

Problem-solving
There is hardly any problem-solving in the Draft. Is this a result of rushing, or did the writers not see the applicability of the material? Problem-solving, both of practical problems and more abstract mathematical problems, has been the heart and soul of the three NSW calculus courses for decades. It is in this problem-solving that one sees most clearly the key insight of the NSW courses — the constant interchange between an algebraic and geometric view of the same problem.

Recommendation 15 for NSW

- Solving practical and theoretical problems should be given far more prominence.
- The motivation of these problems should be to reinforce the constant interchange between an algebraic and geometric view of each part of the syllabus.

The unity of the courses and their examinations

The courses should be unified in a way that reflects the unity of the discipline of mathematics, in which everything that one knows fits together as a coherent whole.

Many things work against this unity at the moment in the Draft Course C. The principal cause is the disorganisation of calculus and its applications, but there is also the lack of a proper review and extension of Year 10 material, the failure to use graphs as an organising principle, the lack of a coherent view of polynomials as a generalisation of quadratics, the failure to develop counting systematically, and the omission of the normal distribution function.

Unity of the courses should be reflected in the approaches that the syllabuses use to build the mathematics in the students' minds. There is little evidence in the Draft of a coherent view of how the student is to assimilate the material.

As remarked above, the 3 unit course should first be conceived as a unity. Then the 2 unit course can be cut from it in such a way that the restricted material is still coherent within itself. This can be achieved by making the integral, and the special functions — logarithmic, exponential and trigonometric — the backbone of both courses.

Examinations are not mentioned in the Draft, which is unsatisfactory in a document going out to teachers for comment. It is vital for the coherence of the courses in the students' and the teachers' minds that in each course, both years' work is examined at the final examinations. It is also vital that if 3 unit students are to be examined by two successive examinations, then all the 2 unit material must potentially be examinable in the 3 unit examination. This has traditionally been done in NSW by the addition of a topic 'Harder problems on the 2 unit course' to the 3 unit course, which means, for example, that a curve-sketching question can consider a function that is too hard for a 2 unit examination.

Recommendation 16 for NSW

- Each of the 2 unit and 3 unit courses should have its own unity, in which everything relates to everything else, and in which theorems are proven as far as is reasonable for the candidature. In particular, the derivative, the integral, the logarithmic and exponential functions, and the trigonometric functions, should form the backbone of each course.
- The examinations for the 2 unit course, and for the 3 unit course, must each examine the whole two-year course in order to preserve this unity.
- In order to preserve the unity of all 3 units, the 3 unit course should have a section within it involving 'Harder questions on the 2 unit course', so that everything in both years of both courses is open to the 3 unit examination.

Part 3: Reconstructing Course D coherently

In these comments, I am writing to the assumption that the eventual Course D will be a 1 unit extension of the 3 unit course that I have proposed. With the future of NSW's 3 unit calculus course unresolved, and the status of ACARA's K–10 curriculum also unresolved, there seems little point in giving too many details at this stage of what the top calculus course should look like, and I will only give general outlines.

Unity of all 4 units should be the guiding concern of this course. NSW's present Year 12 Extension 2 course does an excellent job here. The course is dramatic and exciting, and by the end of it, all sorts of unexpected questions can be asked that bring together material from far-flung places in the three calculus courses. Even though this is only school mathematics, there is a genuine feeling of the extraordinary way in which mathematics is about solving problems by drawing together all available knowledge. I can confidently say to my Extension 2 classes that their course has given them an experience of the true nature of mathematics as a totally unified discipline.

The coherence of the current NSW Extension 2 course comes from extending the Extension 1 course in two contrasting directions:

- Systematic integration — roughly integration of functions built from degree 2 elements — is completed early in the course by adding the primitives of $\dfrac{1}{\sqrt{x^2+1}}$ and $\dfrac{1}{\sqrt{x^2-1}}$, the t-formulae and other trigonometric substitutions, partial fractions, integration by parts, and reduction formulae. These integrals then become tools to be used in the rest of the course, where they are applied to more complicated motion problems, to volumes by slicing, and to the establishing of inequalities arising from definite integrals. The vector approaches to motion introduced in projectiles of Extension 1 are further developed to handle problems in circular motion.
- Complex numbers are developed, which allows the full factoring of quadratics and a much fuller account of polynomials in the context of the fundamental theorem of algebra (stated, but unproven) and the binomial theorem. Complex numbers are combined with trigonometry in de Moivre's theorem and used to establish more advanced trigonometric identities. The use of vectors in the Argand diagram means that the course develops elementary ideas about free vectors.

The possibilities thus opened for imaginative problem-solving are extraordinary, and can only be properly grasped by looking over the HSC examination papers from the last few decades. The implications of this course continue to evolve in the hands of a sequence of much-appreciated academics, who use their scholarship to contribute new and interesting problems each year, while at the same time listening carefully to teachers about the pragmatics of students' abilities.

In contrast to the NSW Extension 2 course, the Draft Course D is incoherent and disorganised. It scatters its approach in irrelevant topics like 'graph theory' and two-dimensional matrices. At the same time, despite its greater length, it fails to provide a proper approach to systematic integration and its applications, or to complex numbers. It also seems to have little awareness of the possibilities of problem-solving inherent in the material.

Calculus

It is a great disappointment to see systematic integration consigned to the end of the course, and part of an Option, when it should be the backbone of the course, as explained above. ACARA is presumably envisaging that the long-standing tradition of calculus in the NSW courses should disappear.

Complex numbers

The treatment of complex numbers is disappointing in comparison to the NSW course. My remarks are deliberately tentative because the Draft's dotpoints do not make intentions clear.

- There seems to be no clear vector interpretation of the difference of two complex numbers.
- There seems to be no coherent relationship between the complex plane and circle geometry, special quadrilaterals and intercepts.
- Apart from multiplication by i, there seems to be no mention of rotation in the complex plane by multiplying by $\cos\theta + i\sin\theta$.
- There seems to be no clear relationship between complex roots and the factoring of $x^n - a^n$.
- There seems to be no use of de Moivre's theorem to obtain trigonometric identities.
- There seems to be no coherent relationship between complex numbers and the sums of products of roots of polynomials.
- There seems to be no statement of the fundamental theorem of algebra.
- There seems to be no clear relationship between complex numbers and the binomial theorem.
- There is no reason to restrict the coefficients of polynomials to real numbers.

It would be a dull course in complex numbers if these things are not remedied.

One detail needs mention. The notation e^{ix} for $\cos x + i\sin x$ is totally inappropriate at school. First, there is no way to justify the notation, because the unification of complex numbers and calculus lies a long way ahead in university mathematics. Secondly, the index laws will no longer work, because, for example, in the absence of any concept of 'multi-valued functions',

$$(e^{ix})^2 = e^{2ix} \text{ is true,} \qquad \text{but} \qquad (e^{ix})^{\frac{1}{2}} = e^{\frac{1}{2}ix} \text{ is false.}$$

Interestingly, when the draft states de Moivre's theorem as $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, it omits the qualification, 'where n is an integer', which is necessary at school. I do not use the cis θ notation in my own classes much, because even very able students constantly lose the intuition that 'cis' stands for '<u>C</u>os + <u>i</u> <u>S</u>in' — I usually write out $\cos\theta + i\sin\theta$ each time. But the cis θ notation is far preferable to $e^{i\theta}$, which would just confuse everything and make the presentation of de Moivre's theorem as a theorem very difficult. School is not the place to present disconnected introductory material to a university course.

Vectors

In contrast to matrices, which require too much abstract machinery, vectors are intuitive and are a good topic for school. An abstract course in vectors here or in 3 unit is an attractive option.

The Draft's proposed vectors topic seems reasonable, but characteristically it is not integrated with the rest of the course. The three current applications of vectors in NSW should continue, because they each provide excellent examples of how vectors make things simpler. NSW presently deals with resolved vectors with projectiles (Extension 1) and circular motion (Extension 2), and with free vectors in complex numbers (Extension 2). I have recommended that projectiles, which are mentioned in the Draft, go into the 3 unit course. The Draft mentions neither circular motion nor vector subtraction specifically, although they could be inferred from some vague dotpoints.

Matrices

In contrast to Course C, there is an attempt here to describe a matrix as a movement in two-dimensional space, but the material is unattractive for several reasons.

- Matrices precede, and are unrelated to, the discussion of vectors, which are what matrices act on. There is, understandably, no discussion of linear independence or spanning, so not much is achieved. As I remarked earlier, matrices as transformations are too hard for school.
- The treatment of matrices here seems unrelated to the treatment of matrices in the Draft Course C or to the solution of simultaneous equations.
- The transformations described can all be effected using complex numbers. Multiplication by $\cos\theta + i\sin\theta$ is used to wonderful effect by the Extension 2 examiners in NSW.
- The item on two simultaneous equations, as written here, seems completely misplaced because it is straightforward Year 10 material, covered again in the opening review in 2 unit.
- Nothing of consequence elsewhere in the course is achieved. One could possibly apply 2×2 rotation matrices to the task of classifying equations of conics, or to rotating graphs in general, but there are better things to do.
- The geometric interpretations of matrix product and determinant are transparently 'Lectures 1 and 2' of a university course, rather than being part of a coherent school course.

Graph theory

Graph theory also does not belong at school, apart from possible rainy-day extension material in Years 7–10. It cannot be integrated with mainstream calculus and algebra, its applications are not evident at school, and it is too unsystematic in its development.

More generally, discrete mathematics is always difficult, because it does not flow and develop systematically in the way calculus does, and it often appears, wrongly, to students to be a bag of unrelated tricks. Some discrete topics, however, are naturally a part of school mathematics — APs and GPs, mathematical induction, ${}^n\mathrm{C}_r$ and counting, discrete probability, and analytic approaches to sequences through definite integrals. These topics can be integrated imaginatively with other topics despite the fact that they are notoriously difficult for school students even at the top level, because they are clearly motivated by problems arising in school calculus and algebra. They provide quite sufficient training in discrete ideas, and the addition of graph theory serves no useful purpose within the school curriculum. The potential for interest in school mathematics does not lie in bits and fragments like this, but in the coherence of the whole course.

Recommendation 17 for NSW

- The final calculus unit should aim to produce a more extended, but still coherent 4 units of Mathematics. This unity should be based on two principles:
  - Systematic integration, using the primitives of $\dfrac{1}{\sqrt{x^2+1}}$ and $\dfrac{1}{\sqrt{x^2-1}}$, the t-formulae and other trigonometric substitutions, partial fractions, integration by parts, and reduction formulae, should be developed early, where it can then be applied to complicated motion problems, to volumes by slicing, and to the establishing of inequalities arising from definite integrals.
  - Complex numbers provide a new coherence to algebra, and can be unified with Euclidean geometry, vector geometry, transformations of the coordinate plane, factoring of polynomials, particularly $x^n - a^n$, the binomial theorem, and trigonometric identities.
- Matrices and graph theory should not be included in the course.
- Whether or not abstract vectors remain, particular attention should be paid to the free vectors as they arise in the Argand diagram through subtraction, and to the resolved vectors as they occur in circular motion (and in projectiles in 3 unit).

Options and statistics

It will be a Herculean effort to introduce the binomial distribution, the normal distribution, and introductory statistical inference into NSW school mathematics, and it would be most unwise to add any more statistics to the senior school curriculum at this stage. It will take quite a few years of intensive teacher-training for the requisite understanding of these three topics to spread through the mathematics-teaching community in NSW.

Again, there is the impression of the universities wanting schools to do their introductory work, rather than letting students learn a coherent and exciting school course in mathematics.

Options are generally not a good thing in school mathematics syllabuses, because the content of an option cannot be used elsewhere in the course, resulting in fragmentation and disunity. Moreover, universities will have to begin at the common knowledge of all their students, and so will never be able to rely on anything that it is in an option. The other two Option topics should not be options at all — as remarked above, systematic integration and its applications should be the backbone of this course, and vectors and dynamics should constitute one of the central applications of calculus.

I suspect, however, that the differing situations in States is the reason for these Options, so I will not make formal recommendations along these lines.

Part 4: Conclusion

ACARA's senior Drafts are not in a good state, and should not have been issued for public comment in their present form. Nor should there be any serious discussion of senior courses until the K–10 courses are in place, properly adapted for NSW, and well accepted. To summarise the potential damage to NSW of the senior Drafts:

- The loss of the 1 unit Extension courses in humanities would destroy a long-standing NSW tradition of excellence in the humanities.
- Forcing NSW students who want to excel in either Mathematics and English to take 4 units of the subject in Year 11 will narrow the breadth of the senior curriculum, to the detriment of other humanities, particularly languages. It will also force subject-choice decisions on students before they have enough experience of their own strengths and interests, and before teachers can confidently advise them.
- The loss of 3 units of Mathematics will lead to a serious 'dumbing down' of mathematics in NSW schools, and will lead in turn to a serious deterioration of standards in science and economics in NSW universities.
- The Draft Course C is too hard for the present 2 unit candidature in NSW, and yet is inadequate preparation for university mathematics in comparison to the present NSW Extension 1 course.
- Judged by the standards of the present NSW 2 unit and Extension 1 calculus courses, the Draft Course C is incoherent in its order of topics and in its presentation of them, particularly as regards calculus itself and the use of graphs. The Draft seems indifferent to the way in which school mathematics is assimilated by students in the classroom.
- Judged by the standard of the NSW Extension 2 course, the Draft Course D is also incoherent. The Draft makes excursions into inappropriate topics, and it fails to present a clear, unified account of either calculus or algebra.

Whereas the K–10 situation could possibly be remedied with some flexibility and understanding towards NSW's proven structures, it is not at all clear that anything can be salvaged from the current Years 11–12 proposals. ACARA seems to have no coherent view of senior mathematics, no coherent view of the curriculum, and no coherent view of the child.

I have tried nevertheless to make a series of sensible proposals that could be used as a basis for sorting out the situation. To summarise the essence of those proposals:

- NSW must be allowed to retain its structure of 1 unit Extension courses in humanities, English and Mathematics, and not be required to offer 4 units of English or Mathematics in Year 11.
- The 3 unit Mathematics course should continue to be regarded as the entrance requirement for university mathematics in NSW, and should be written first, with coherence and appropriate rigour. A coherent 2 unit Mathematics course can then be cut from it in such a way that a less able candidature is accommodated, but bridging to 3 unit possible.
- The final 1 unit extension course should be a Year 12 course only, written so that all 4 units form a unified course based on systematic integration and complex numbers.
- Each calculus examination in NSW should examine the full two years' work, and should also contain harder questions on the less demanding courses.
- The courses mentioned above could be written by ACARA, or by the NSW Board of Studies within a more flexible structure than presently envisaged.

If these things cannot be achieved through the National Curriculum structure, then NSW has no option but to withdraw from the process. Uniformity of syllabuses across Australia cannot be imposed at the expense of destroying the great strengths that NSW has at present in its structures and syllabus content. The fragmented approaches of ACARA's present proposals do not compare with the decades of experience and wisdom of the Board of Studies, of the DET, of academics, and above all, of the State's teachers.

Our State's education system is excellent, and in particular the calculus courses continue to go from strength to strength in the hands of enthusiastic teachers and students and the guidance and intiative of our examiners. Our universities have a good supply of students well prepared in mathematics, a situation quite unlike any other State. We look to the NSW Board of Studies to be the guardian of all that is best in NSW education.

Yours sincerely,

Dr Bill Pender
Australian Mathematical Sciences Institute
(These are the writer's personal views and do not represent an opinion of AMSI.)

Cc: Mr Peter Osland
Chief Inspector of Mathematics
Office of the Board of Studies NSW
GPO Box 5300
Sydney NSW 2001

Mr Rob Randall
General Manager Curriculum
ACARA
Level 10, 255 Pitt Street
Sydney NSW 2000

ACAAA DRAFT

Table 1: Scope of the Australian Curriculum K-10

Notes

a) The indicative hours are developed to guide curriculum writers; decisions about the actual organisation and delivery of curriculum, including opportunities for integration are best taken at the school level.

b) Hours / year; assuming total of 1000 hours of teaching time each year (25 hours of teaching time each week; 40 weeks / year).

c) indicates that study will be optional ie the student will decide (but learning will be based on Australian Curriculum)

Learning Area	Subject	Year K	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10
English		280 hrs	280 hrs	280 hrs	240 hrs	240 hrs	240 hrs	240 hrs	160 hrs[1]	160 hrs	160 hrs	160 hrs
Mathematics		200 hrs	200 hrs	200 hrs	200 hrs	200 hrs	200 hrs	200 hrs	160 hrs[2]	160 hrs	160 hrs	160 hrs
Science		40 hrs	40 hrs	40 hrs	80 hrs	80 hrs	80 hrs	80 hrs	120 hrs	160 hrs	160 hrs	160 hrs
Humanities and social sciences	History	20 hrs	20 hrs	20 hrs	40 hrs	40 hrs	40 hrs	40 hrs	80 hrs	80 hrs	80 hrs	80 hrs
	Geography	20 hrs	20 hrs	20 hrs	40 hrs	40 hrs	40 hrs	40 hrs	40 hrs	40 hrs	80 hrs*	
	Economics, Business[3]	-	-	-	-	-	20 hrs	20 hrs	20 hrs	20 hrs	80 hrs*	
	Civics and Citizenship[4]	-	-	-	-	-	20 hrs	20 hrs	20 hrs	20 hrs	40 hrs*	
The Arts[5]		160 hrs			160 hrs		160 hrs		160 hrs		160 hrs[6]*	
Health & PE[7]		80 hrs	80 hrs	80 hrs	80 hrs	80 hrs	80 hrs	80 hrs	80 hrs	80 hrs	80 hrs	80 hrs
Languages		500 hrs							150 hrs		150 hrs*	
Technologies	Design & Tech	-	-	-	20 hrs	20 hrs	20 hrs	20 hrs	40 hrs	40 hrs	80 hrs*	
	ICT[8]	-	-	-	20 hrs	20 hrs	20 hrs	20 hrs	20 hrs	20 hrs	80 hrs*	
TOTAL		765	765	765	870	870	910	910	895	935	640	640

[1] Reduced from 200 hours in current Curriculum Design specification
[2] Reduced from 200 hours in current Curriculum Design specification
[3] Including consumer and financial literacy
[4] This will build on the position outlined in the Statement of Learning for Civics and Citizenship, with particular attention given to the political and engagement aspects of the statement.
[5] The Arts – Music, Visual Arts, Drama, Dance and Media
[6] In Years 9 and 10 students will choose which specific art form(s) they will study
[7] Does not fully account for the requirement for physical activity, although H&PE will contribute to that requirement
[8] Specification of this area will take account of the national statement of learning and what is included in the ICT general capability.