1916 New South Wales Leaving Certificate

Mathematics Honours Papers

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

New South Wales Department of Education

PAPER I

Time Allowed: 3 Hours

1. Prove that the radical axes of three circles taken in pairs are concurrent. Use this theorem to obtain a construction for a circle through two given points which shall touch a given circle. Prove the validity of your construction, and show that there are two such circles.

2. Prove that the inverse of a circle with regard to a point upon its circumference is a straight line, and with regard to a point not on its circumference is another circle. Two parallel straight lines lie on the same side of a given point P. A circle is described to pass through P and to touch the more remote of the given lines at Z, cutting the other at X and Y. Prove that the angles XPZ and YPZ are equal. Draw the system of lines and circles which you obtain by inverting the system with regard to P, and enunciate the inverse proposition.

3. A, B, C, D are the four corners of a face of a cube, taken in order, and $CDEF$ is another face of the same cube, AED and BCF being the faces perpendicular to $CDEF$. Find the sizes of the angles AEF, AFE, DFE, and AFD. Also find the length of the perpendicular from E on ADF.

4. Prove the two following theorems:-

(i) Of the three plane angle which form a trihedral angle, any two are together greater than the third.

(ii) The sum of the plane angles of a convex solid angle is less than four right angles.

5. With the usual notation prove that in the parabola

$$PN^2 = 4AS.AN$$

SL is the semi-latus rectum of a parabola, whose vertex is A, and X is the point where the axis meets the directrix. A second parabola has S for its focus and X for its vertex. The line LX meets the directrix of this second parbola in Q and QP is drawn parallel to the axis to meet the same parbola in P. Show that the ordinate of P bisects AX.

6. Prove, by orthogonal projection of a circle, the following properties of an ellipse:-

(i) The tangents at the ends of any chord meet on the diameter which bisects the chord.

(ii) If a diameter CP meets a chord in V, and the tangents at its extremities in T, then $CV.CT = CP^2$.

(iii) Lines drawn through any point of an ellipse to the extremities of any diameter meet the conjugate diameter CD in M and N. Show that

$$CM.CN = CD^2.$$

7. Prove that

$$\cos A = 4\cos^3 \tfrac{A}{3} - 3\cos \tfrac{A}{3},$$
$$\sin A = 3\sin \tfrac{A}{3} - 4\sin^3 \tfrac{A}{3}.$$

(i) Find all the possible values of $\cos \frac{A}{3}$.

(ii) Find all the possible values of $\sin \frac{A}{3}$.

(iii) Find how many values there are in general for $\sin \frac{A}{3}$ and for $\cos \frac{A}{3}$.

8. Prove that if $\cos 2\theta = \cos 2\alpha \cos 2\beta$, then $\sqrt{(\cot^2 \alpha + \cot^2 \beta)} = \frac{\sin \theta}{\sin \alpha \sin \beta}$. If $\alpha = 62°$ and $\beta = 54°$, find all the possible values of θ which satisfy the former of these equations, and show that their sines have the same absolute value.

The station A is due south, and the station B due east, of O. The altitudes of an aeroplane vertically above O are $62°$ and $54°$ from A and B respectively. If AB is 1 mile, find the height of the aeroplane above O.

9. R is the radius of the circumcircle and r that of the inscribed circle of the triangle ABC. Prove that

$$r = 4R \sin \tfrac{A}{2} \sin \tfrac{B}{2} \sin \tfrac{C}{2}.$$

The inscribed circle touches the sides of the triangle ABC in D, E, F. Show that the area of the triangle DEF is given by $Rr \sin A \sin B \sin C$.

10. Prove that, for real values of $n, \cos n\theta + i \sin n\theta$ is the value, or one of the values, of $(\cos\theta + i\sin\theta)^n$, and in cases in which there is more than one value give all the values of $(\cos\theta + i\sin\theta)^n$. Find a linear factor and two quadratic factors of $x^5 + 1$.

11. What is a radian? Show that π radians are equal to 2 right angles. Verify that between 108° and 109° there is an angle whose sine is half its circular measure. Show, graphically or otherwise, that there is only one positive angle with this property.

12. Sum to n terms the following series:-

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \cdots$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \cdots$

(iii) $\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \cdots$

(iv) $\sin\theta + 2\sin 2\theta + 3\sin 3\theta + \cdots$

Also show that the series

$$\cos\theta + x\cos 2\theta + x^2\cos 3\theta + \cdots$$

is convergent when $-1 < x < 1$, and that its sum is

$$\frac{\cos\theta - x}{1 - 2x\cos\theta + x^2}$$

END OF PAPER I

PAPER II BEGINS ON THE NEXT PAGE

PAPER II

Time Allowed: 3 Hours

1. What is the ratio test for the convergence of the series

$$u_0 + u_1 + u_2 + \cdots \quad ?$$

Establish the theorem in the form in which you have stated it. Discuss the convergence of the series

(i) $1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots \qquad (-1 < x < 1)$

(ii) $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (-1 < x < 1).$

Show that the second series is convergent when $x = 1$.

2. Apply the Binomial Theorem to show that

$$\left(\tfrac{3}{4}\right)^{\frac{4}{5}} = \cdot 7944$$

correct to four decimals. Establish carefully that your approximation is as stated.

3. Solve the following question in permutations and combinations:-

(a) Six persons $A, B, C, D, E,$ and F are to address a meeting. In how many ways can they take turns so that

(i) A, B are consecutive speakers;

(ii) A, B are consecutive speakers, and C, D are consecutive speakers?

The order of A and B respectively, or C and D respectively, is not to be considered.

(b) There are six persons from whom a game of tennis, two on each side, is to be made up. How many different matches could be arranged, a change in either pair giving a different match?

4. Establish the identity

$$\frac{x}{x^5+1} = -\frac{1}{5(x+1)} - \frac{2}{5}\sum_{r=1}^{2} \frac{x\cos 2(2r-1)\frac{\pi}{5} - \cos(2r-1)\frac{\pi}{5}}{x^2 - 2x\cos(2r-1)\frac{\pi}{5} + 1}$$

5. Assuming that

$$\log_e(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots, \quad (-1 < x < 1)$$

show that

$$\log_e\left(\tfrac{1+x}{1-x}\right) = 2\bigl(x + \tfrac{x^3}{3} + \cdots\bigr),$$

when x is numerically less than unity. Also show that the error in taking

$$2\bigl(x + \tfrac{x^3}{3} + \cdots + \tfrac{x^{2n-1}}{2n-1}\bigr)$$

for $\log_e\left(\frac{1+x}{1-x}\right)$ is less in absolute value than

$$\tfrac{2|x|^{2n+1}}{(2n+1)(1-x^2)},$$

with the same range for x.

6. Use the second part of the preceding question to find $\log_e 2$ and $\log_e 10$ correct to seven places.

7. Find the condition that the three points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3)$$

may be collinear. A is the point $(a, 0)$, B the point $(0, b)$ and C the point (b, a). AC meets the axis of y in F, and BC meets the axis of x in E. Find the co-ordinates of E and F; also show, by analytical geometry, that the middle points of EF, OC and AB are collinear.

8. Find the equation of the line through (x_0, y_0) perpendicular to the line $ax + by + c = 0$. Show that the distance from (x_0, y_0) to the point of intersection of these two lines is

$$\frac{(ax_0 + by_0 + c)}{\sqrt{(a^2 + b^2)}}.$$

Verify that the point $(3\sqrt{5} - 7, \sqrt{5} + 1)$ is equally distant from the three lines

$$\left.\begin{aligned} x + 2y - 10 &= 0 \\ 4x + 3y \qquad &= 0 \\ 2x - \ \ y \qquad &= 0 \end{aligned}\right\}.$$

9. Prove that the equation of the tangent at the point $(c\cos\theta, c\sin\theta)$ on the circle $x^2 + y^2 = c^2$ is

$$x\cos\theta + y\sin\theta = c.$$

Show that if θ satisfies the equation

$$a\cos\theta + b\sin\theta = c,$$

the tangent at $(c\cos\theta, c\sin\theta)$ passes through the point (a, b), and thus obtain a graphical solution of this equation, when $c^2 < a^2 + b^2$. By means of this figure, show that, if α, β are the roots of the equation, then

$$\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{c} = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{a} = \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{b}$$

10. Find the centre and radius of each of the circles:-

$$\left.\begin{aligned} x^2 + y^2 - 4x - 2y + 1 = 0 \\ x^2 + y^2 - 5x + y - 6 = 0 \end{aligned}\right\}.$$

Draw the circles and thus obtain graphically the values of (x, y) which satisfy both equations. Find the equation of the common chord of the circles, and show that it cuts the axis of y at a distance $\frac{7}{3}$ from the origin.

11. Find, from the definition of the differential coefficient, the value of $\frac{dy}{dx}$, when $y = (x-1)(x-2)$. Show that the gradients of this curve at the points where $x = 1, x = \frac{3}{2}$ and $x = 2$ are respectively $-1, 0$ and $+1$. Obtain the equations of the tangents at these points.

12. Prove the rule for differentiating the product of two functions. Differentiate the expression $(ax+b)^2(cx+d)^2$ in two ways, first as a product, then after multiplying it out. Show that the gradient of the curve

$$y = (ax+b)^2(cx+d)^2$$

is zero, when x has the values $-\frac{b}{a}, -\frac{d}{c}$, and $-\frac{ad+bc}{2ac}$.

13. Find an expression for the area of a parabola from the vertex to the latus rectum. Two parabolas have the same focus and axis, and their concavities are turned in the same direction. If the vertices are distant a and b from the focus, show that the areas cut from each by the latus rectum are in the ratio $a^2 : b^2$.

14. Prove that the volume V of a solid of revolution, the axis of y being the axis of the solid, satisfies the equation

$$\frac{dV}{dy} = \pi x^2.$$

A hemispherical bowl of radius a ft. stands with its axis vertical, and a solid cone of semi-vertical angle α is placed in the bowl with its vertex at the lowest point and its axis also vertical. If the space between the cone and the bowl is to be filled with water to a depth h ft., show that $\pi h^2(a - \frac{h}{3}\sec^2\alpha)$ cub. ft. will be required.