MORIAH COLLEGE

Year 12

MATHEMATICS

Extension 2

Date: **Wednesday 8th August, 2001**

Time Allowed: 3 hours, plus 5 minutes reading time.

Examiners: J. Taylor

Instructions:

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 6
- Board approved calculators may be used.
- Answer each question in a SEPARATE writing booklet.
- You may ask for extra Writing Booklets, if you need them.

Question 1 (15 marks)

a) Find

i) $$\int \frac{\cos^{-1}\frac{2x}{3}}{\sqrt{9-4x^2}}\,dx$$ **2**

ii) $$\int x^2 \tan^{-1}\frac{x}{2}\,dx$$ **3**

iii) $$\int \frac{dx}{\sqrt{2x^2+3x}}$$ **4**

b) If **6**

$$I_n = \int_0^1 x^n (1-x)^{\frac{1}{2}}\,dx \qquad (n > 0)$$

prove that

$$I_n = \left(\frac{2n}{2n+3}\right) I_{n-1}$$

and evaluate I_2.

Question 2 (15 marks)

a) Sketch the region in the Argand Plane consisting of those points z for which

$|z+3-i|>4$ intersects with $-\dfrac{3\pi}{8}<\arg z\le-\dfrac{3\pi}{4}$ **3**

b) Given that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

i) Find, in terms of π, an approximations for $e^{i\pi}$ in the form $x + iy$ using the above four terms of the series. **1**

ii) Use your calculator where necessary to plot this approximation on an Argand diagram **2**

iii) If z is any complex number, prove that both **4**

$z^n + \bar{z}^n$

and

$e^z + e^{\bar{z}}$

are pure real numbers

c) i) Find the five fifth roots of unity. **2**

ii) If $\omega = \text{cis}\dfrac{2\pi}{5}$, show that $1+\omega+\omega^2+\omega^3+\omega^4=0$ **1**

iii) Show that $z_1=\omega+\omega^4$ and $z_2=\omega^2+\omega^3$ are the roots of the equation

$z^2+z-1=0$ **2**

Question 3 (15 marks)

a) Sketch $y = \sqrt{\cos 3x}$ in the domain: $-\pi \le x \le \pi$ **2**

b) The rational function $f(x)$ is defined

$$f(x) = \frac{x - p}{(x - q)(x - r)}$$

i) Write $f(x)$ as the sum of two partial fractions. **2**

ii) If p lies between q and r, use a neat sketch to explain why $f(x)$ can assume all real values. **2**

iii) If p does not lie between q and r, use a neat sketch to explain why $f(x)$ *cannot* assume all real values. **2**

iv) Show that in either case, the gradient of the tangent at the midpoint of the interval between q and r is independent of p **3**

c) If x, y are positive integers such that $x - y > 1$, then prove that

$$x! + y! > (x - 1)! + (y + 1)!$$ **3**

Question 4 (15 marks)

a) For the ellipse $\frac{y^2}{50}+\frac{x^2}{32}=1$, find

i) the eccentricity, **2**

ii) the coordinates of the foci S and S'. 1

b) Explain why $\frac{x^2}{\lambda-23}+\frac{y^2}{5-\lambda}=1$ cannot represent the equation of an ellipse. **1**

c) Normals to the ellipse $4x^2+9y^2=36$ at points $P(3\cos\alpha, 2\sin\alpha)$ and $Q(3\cos\beta, 2\sin\beta)$ are at right angles to each other. Show that

i) the gradient of the normal at P is $\frac{3\sin\alpha}{2\cos\alpha}$, **2**

ii) $4\cot\alpha\cot\beta=-9$. **2**

d) $P\left(5p,\frac{5}{p}\right), p>0$ and $Q\left(5q,\frac{5}{q}\right), q>0$ are two points on the hyperbola, *H*, $xy=25$.

i) Derive the equation of the chord *PQ*, **2**

ii)` State the equations of the tangents at *P* and *Q*, **1**

iii) If the tangents at *P* and *Q* intersect at *R*, find the co-ordinates of *R*. **2**

iv) If the secant *PQ* passes through the point $S(15,0)$, find the locus of *R*. **2**

Question 5 (15 marks)

a) When $x^3 - kx^2 - 10kx + 25$ is divided by $x - 2$ the remainder is 9. Find the value of *k*. **2**

b) A polynomial function is $P(x) = x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$. Factorise $P(x)$ over the field of

i) real numbers, **2**

ii) complex numbers. **1**

c) The equation $x^5 - 5x^4 - x^3 + 3x^2 + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equations with roots

i) $\frac{1}{\alpha} + 2, \frac{1}{\beta} + 2, \frac{1}{\gamma} + 2, \frac{1}{\delta} + 2$, **2**

ii) $\alpha^2 - 1, \beta^2 - 1, \gamma^2 - 1, \delta^2 - 1$ **3**

d) $\phi(x)$ is a polynomial of degree 5 such that $\phi(x) - 1$ is divisible by $(x-1)^3$ and $\phi(x)$ itself is divisible by x^3. Derive an expression for $\phi(x)$. **5**

Question 6 (15 marks)

a) i) Prove that $2\pi \int_1^5 x\sqrt{4-(x-3)^2}\,dx = 12\pi^2$ **3**

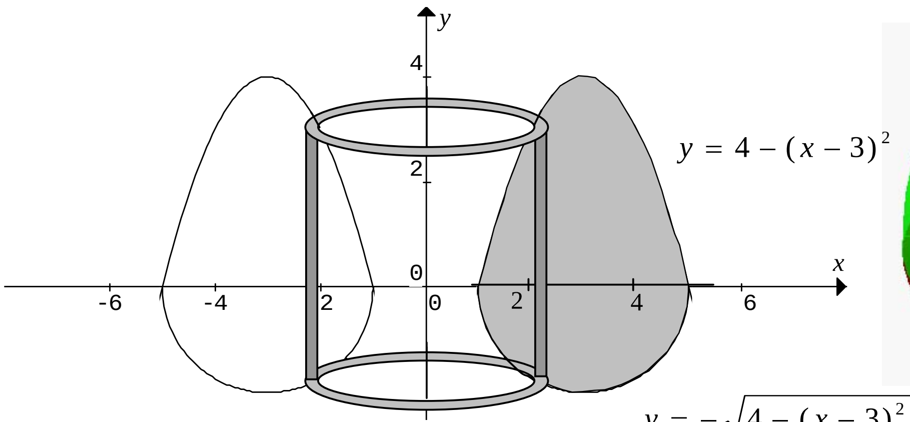


Figure 1

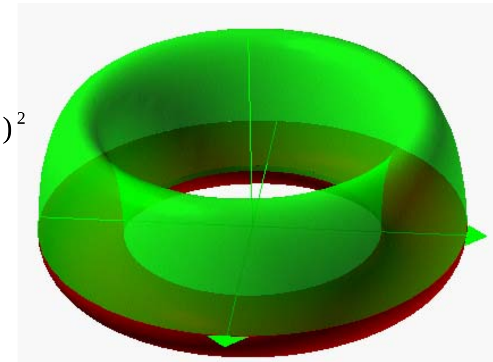


Figure 2

ii) The solid in fig.2 is formed by rotating about the *y*-axis the area bounded by the parabola $y = 4-\left(x-3\right)^2$ and the semi-circle $y = -\sqrt{4-\left(x-3\right)^2}$.

Use the method of cylindrical shells to calculate the volume generated. **3**

Question 6 b) is on the next page.

Question 6

b) A circle in a horizontal *x*-*y* plane has centre *O*, radius *R* units . At each point *P*(*x*, *y*) on this circle, another circle is constructed perpendicular to the original circle in the plane containing the radius at that point. The radius *r* of such a circle (see the shaded circle, figure 1) is given by

$r = xy.$

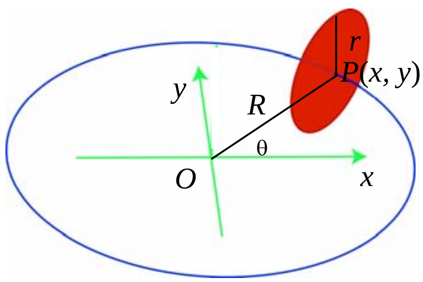


Figure 1

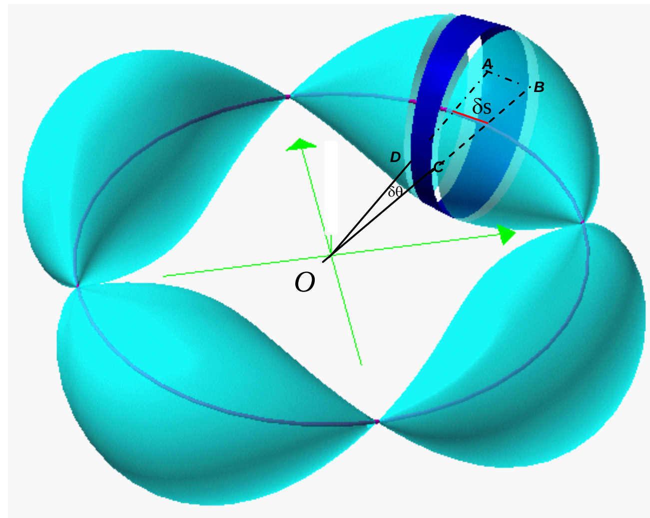


Figure 2

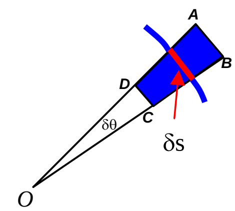
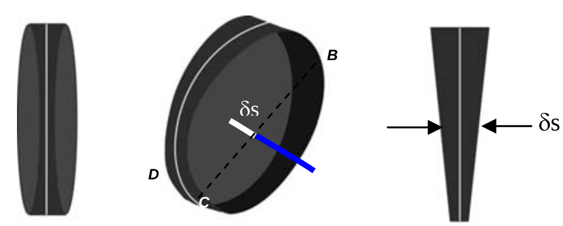


Figure 3

i) If $\angle POx = \theta$, prove that $r = \dfrac{R^2 \sin 2\theta}{2}$ **2**

As *P* moves around the horizontal circle, the vertical circles will form a surface drawn in figure 2.

A section is taken in the first quadrant by slicing the figure with two vertical planes from the centre of the horizontal circle. This section can be approximated by the wedge-shaped solid in figure 4:



Front *Perspective* *Top*

Figure 4

ii) If the thickness of the section at the centre is the straight line length δs, explain *briefly* why the volume δV of the section in figure 4 is $\delta V = \pi r^2 \delta s$. **1**

iii) If $\delta\theta$ is the angle between the radii used to slice the figure(see figure 3), prove that **3**

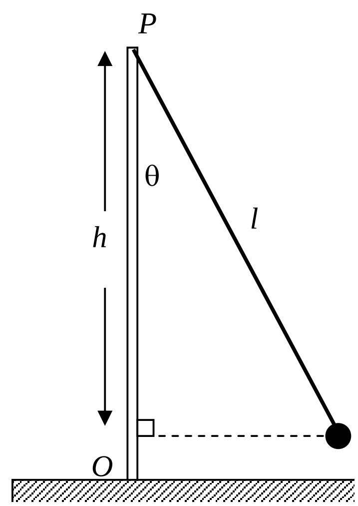
a) $\delta V \approx R\pi r^2 \delta\theta$

b) $$V = \frac{R^5}{4}\pi \lim_{\delta\theta \to 0} \sum_{\theta=0}^{2\pi} \sin^2 2\theta\,\delta\theta$$

iv) Find the volume of the solid. **3**

Question 7 (15 marks)

a) A vertical pole PO of height l units is standing on a flat plane. Suspended from the top P of the pole is a string, also of length l, at the end of which is attached a particle of mass m. The pole begins to rotate so that the mass describes a horizontal circle of radius r units with a uniform angular velocity so that the centre of the circle is h units below P.



The string makes an angle θ with the pole.

i) If the speed of the particle is v, prove that **2**

$$\tan\theta = \frac{v^2}{gr}$$

ii) When the particle is uniformly rotating at a height $h = \frac{l}{2}$ the string suddenly snaps, and the particle travels freely through the air to land at a point Q. **4**

Find the distance OQ in terms of l.

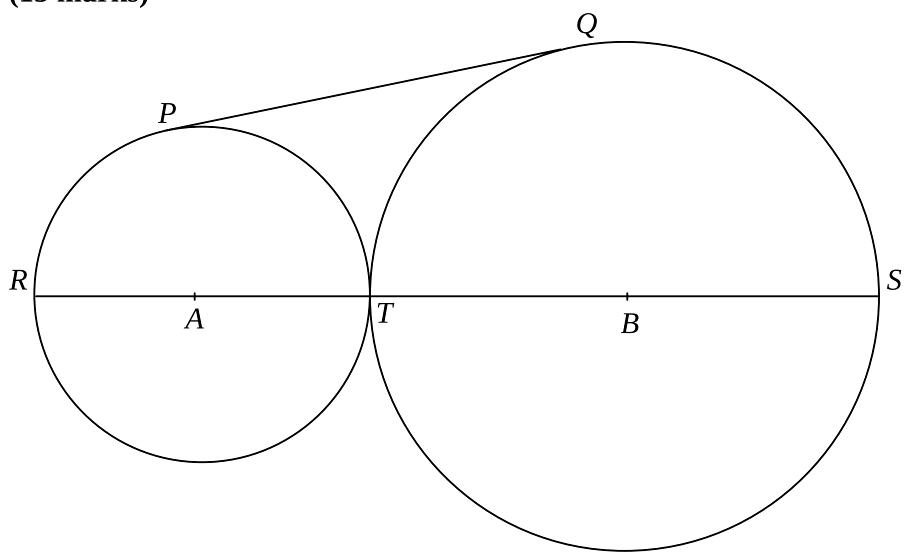
b) The only force acting on a particle moving in a straight line is a resistance $m\lambda(c+v)$ acting in the same line. The mass of the particle is m, its velocity is v, and λ and c are positive constants. The particle starts to move with velocity u (>0) and comes to rest in time T. At time $\frac{1}{2}T$ its velocity is $\frac{1}{4}u$. Show that

i) $c = \frac{1}{8}u$, **6**

ii) at time t, $8\frac{v}{u} = 9e^{-\lambda t} - 1$. **3**

Question 8 (15 marks)

a)



Copy the above diagram into your answer booklet.

In the diagram, circles with centres *A* and *B* touch each other at *T*. *PQ* is a direct common tangent. The line of centres cuts the circles at *R* and *S* as shown.

RP and *SQ* produced meet at *X*.

Prove that $\angle RXS$ is a right angle. **5**

b) A sequence of polynomials (called the *Bernoulli Polynomials*) is defined inductively by the three conditions:

1) $B_0(x) = 1$

2) $B_n'(x) = nB_{n-1}(x)$

3) $\int_0^1 B_n(x)dx = 0 \text{ if } n \geq 1$

i) Prove that $B_1(x) = x - \frac{1}{2}$ **3**

ii) Prove that if $B_n(x+1) - B\ (x) = nx^{n-1}$ and

$$g(x) = B_{n+1}(x+1) - B_{n+1}(x)$$

then

$$g'(x) = (n+1)nx^{n-1}$$

and deduce an expression for $g(x)$. **4**

iii) Prove by induction that

$$B_n(x+1) - B_n(x) = nx^{n-1} \text{ if } n \geq 1.$$ **3**

End of Paper.

STANDARD INTEGRALS

$\int x^n dx \quad = \quad \frac{1}{n+1} x^{n+1}, \; n \neq -1; \;\; x \neq 0, \text{ if } n < 0.$

$\int \frac{1}{x} dx \quad = \quad \ln x, \; x > 0.$

$\int e^{ax} dx \quad = \quad \frac{1}{a} e^{ax}, \; a \neq 0.$

$\int \cos ax dx \quad = \quad \frac{1}{a} \sin ax, \; a \neq 0$

$\int \sin ax dx \quad = \quad -\frac{1}{a} \cos ax, \; a \neq 0$

$\int \sec^2 ax dx \quad = \quad \frac{1}{a} \tan ax, \; a \neq 0$

$\int \sec ax \tan ax dx \quad = \quad \frac{1}{a} \sec ax, \; a \neq 0$

$\int \frac{1}{a^2 + x^2} dx \quad = \quad \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}} dx \quad = \quad \sin^{-1} \frac{x}{a}, \; a \neq 0, \; -a \leq x \leq a$

$\int \frac{1}{\sqrt{x^2 - a^2}} dx \quad = \quad \ln \left\{ x + \sqrt{(x^2 - a^2)} \right\}, \; |x| > |a|$

$\int \frac{1}{\sqrt{x^2 + a^2}} dx \quad = \quad \ln \left\{ x + \sqrt{(x^2 + a^2)} \right\}$

NOTE: $\ln x = \log_e x, \; x > 0.$