Extension to 2002 HSC Ext. 2 Q8a

(v) Use the identity $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ and the result of Question 8 (a) (iii) to prove that

$$\operatorname{cosec}^2\left(\frac{\pi}{2m+1}\right) + \operatorname{cosec}^2\left(\frac{2\pi}{2m+1}\right) + \cdots + \operatorname{cosec}^2\left(\frac{m\pi}{2m+1}\right) = \frac{2m(m+1)}{3}$$

(vi) Use the fact that $\operatorname{cosec}\theta > \dfrac{1}{\theta}$ for $0 < \theta < \dfrac{\pi}{2}$ to show that:

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2}\right)\frac{(2m+1)^2}{4m(m+1)} < \frac{\pi^2}{6}$$

(vii) Deduce that $\frac{1}{(2m+1)^2} < 1 - \frac{6}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2}\right) < \frac{6m+1}{(2m+1)^2}$

(viii) Taking the limit as $m \to \infty$ deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$