Question 8

2002 HSC ME2 p11

(a) (i) $\cos(2m+1)\theta + i\sin(2m+1)\theta = (\cos\theta + i\sin\theta)^{2m+1}$

$RHS = {}^{2m+1}C_0 \cos^{2m+1}\theta (i\sin\theta)^0 + {}^{2m+1}C_1 \cos^{2m}\theta (i\sin\theta)^1 +$

$\boxed{i^k, k=0,1,2,\ldots \text{ alternates from } 1, i, -1, -i, 1, i \text{ etc.}}$ $\ldots + {}^{2m+1}C_{2m} \cos\theta (i\sin\theta)^{2m} +$

${}^{2m+1}C_{2m+1} (\cos\theta)^0 (i\sin\theta)^{2m+1}$, $2m+1$ is odd.

Equating imaginary parts in LHS and RHS,

$$\sin(2m+1)\theta = {}^{2m+1}C_1 \cos^{2m}\theta \sin\theta - {}^{2m+1}C_3 \cos^{2m-2}\theta \sin^3\theta + \ldots + (i)^{2m}\, {}^{2m+1}C_{2m+1} (\cos\theta)^0 \sin^{2m+1}\theta$$

$$(i)^{2m} = (i^2)^m = (-1)^m$$

as required.

(ii) Upon inspection of the RHS in $p(x)$ and the RHS in (i), particularly the last term, we get a vague impression that we can try to divide the whole thing in (i) by $\sin^{m+1}\theta$ to see what the result looks like (provided $\sin^{2m+1}\theta \neq 0$).

$$① — \frac{\sin(2m+1)\theta}{\sin^{2m+1}\theta} = \binom{2m+1}{1}\cot^{2m}\theta - \binom{2m+1}{3}\cot^{2m-2}\theta + \ldots + (-1)^m$$

we write the RHS in cot probably because we see cot in the question ... trial and error.

comparing the RHS to $p(x)$, we see that we can put $x = \cot^2\theta$.
Roots or zeroes of $p(x)$ is when $p(x) = 0$, i.e. RHS in ① $= 0$
This happens when the numerator on the LHS $= 0$

$$\sin(2m+1)\theta = 0$$
$$(2m+1)\theta = n\pi, \quad n \in \mathbb{Z}$$
$$\theta = \frac{n\pi}{2m+1}$$

$\therefore$ roots $= \cot^2\theta = \cot^2\left(\frac{n\pi}{2m+1}\right)$

but unique solutions can be obtained by restricting θ to the first quadrant, $0 < \theta \leq \pi/2$
(we exclude $\theta = 0$ because then $\sin^{2m+1}\theta = 0$)

$\therefore$ $0 < n \leq m + \frac{1}{2}$, but n integer.

$n = k = 1, 2, 3, \ldots, m.$

$\therefore \alpha_k = \cot^2\left(\frac{k\pi}{2m+1}\right)$. α_k's are distinct because for each k the angle $\left(\frac{k\pi}{2m+1}\right)$ is unique and in the first quadrant.

Question 8

(a)(iii) The terms on the LHS are the roots α_k.

sum of roots $= -\dfrac{b}{a}$

$$= \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{(2m+1)!}{(2m+1-3)!\,3!} \times \frac{(2m+1-1)!\,1!}{(2m+1)!}$$

$$= \frac{(2m)(2m-1)}{3!} = \frac{m(2m-1)}{3}$$

(iv)

$$\cot\theta < \frac{1}{\theta}, \quad \cot\theta > 0, \quad \frac{1}{\theta} > 0$$

$$\cot^2\theta < \left(\frac{1}{\theta}\right)^2$$

$$\sum \cot^2\theta < \sum \left(\frac{1}{\theta}\right)^2$$

$$\frac{m(2m-1)}{3} < \sum_{k=1}^{m} \left(\frac{2m+1}{k\pi}\right)^2$$

$$= \frac{(2m+1)^2}{1^2\pi^2} + \frac{(2m+1)^2}{2^2\pi^2} + \dots + \frac{(2m+1)^2}{m^2\pi^2}$$

$$= \frac{(2m+1)^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{m^2}\right)$$

rearranging, $\dfrac{\pi^2}{3} < \left(\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dots + \dfrac{1}{m^2}\right)\dfrac{(2m+1)^2}{m(2m-1)}$

$$\frac{\pi^2}{6} < \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{m^2}\right)\frac{(2m+1)^2}{2m(2m-1)}$$