(Western Region)

2003

TRIAL HIGHER SCHOOL CERTIFICATE

EXAMINATION

Mathematics
Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for every question.

Total marks (120)

- Attempt Questions 1-8
- All questions are of equal value
- Start a fresh sheet of paper for each question.
- Put your name and the question number at the top of each sheet.

Question 1 **Marks**

(a) Evaluate $\displaystyle\int_0^4 \frac{8dx}{16+x^2}$ 2

(b) Find $\displaystyle\int xe^x\,dx$ 2

(c) Find $\displaystyle\int \frac{\sin^3 x\,dx}{\cos^2 x}$ 3

(d) By using the substitution $x = 2\cos\theta$ or otherwise evaluate $\displaystyle\int_{-2}^{0} \sqrt{4-x^2}\,dx$ 4

(e) Given that $\dfrac{4x^2+3x+3}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

(i) find the values of the constants A, B and C 2

(ii) hence, or otherwise find $\displaystyle\int \frac{4x^2+3x+3}{(x+1)(x^2+1)}\,dx$ 2

Question 2 **Marks**

(a) Given the complex numbers

$A = 3 + 4i$ and $B = 2 - 3i$, write the following in the form $x + iy$

(i) $A + B$ 1

(ii) $\frac{A}{B}$ 1

(iii) $\sqrt{A}$ 3

(b) If the complex number $z = 1 + \sqrt{3}i$

(i) Find (α) $|Z|$ 1

(β) $Arg Z$ 1

(ii) Hence write Z in modulus - argument form 1

(iii) By using your answer for part (ii) or otherwise write the complex number Z^4 in the form $x + iy$ 1

(c) The triangle POQ is right angled at O. The length of OQ is twice that of OP. (O is the origin and Q is in the second quadrant). Given that OP represents the complex number $3 + 4i$

(i) Determine the complex number represented by OQ 1

(ii) Determine the complex number represented by QP. 1

(d) Given that $Z = \cos\theta + i\sin\theta$

(i) Show that $Z^n + \frac{1}{Z^n} = 2\cos n\theta$ 1

(ii) Hence by using (i) and binomial expansion, write $\cos^4\theta$ in terms of $\cos n\theta$ (where $n = 2,4$) 3

Question 3 **Marks**

(a) The parabolas $y = x^2 - 3x$ and $x = y^2 - 2y$ intersect at four points:

(i) Show that the x and y coordinates of these points would be the roots of $x^4 - 6x^3 + 7x^2 + 5x = 0$ and $y^4 - 4y^3 + y^2 + 5y = 0$ respectively. **2**

(ii) A quadrilateral is formed by joining these four points. It is known that the coordinates of the balance point of the quadrilateral are given by the average of the coordinates of the vertices. **2**
Without finding the vertices determine the coordinates of the balance point.

(b) The equation $x^3 + 3x^2 - 4x + 5 = 0$ has roots of α, β and γ:

(i) Write down an equation which has roots α^2, β^2 and γ^2 **3**

(ii) Evaluate $\alpha^3 + \beta^3 + \gamma^3$ **3**

(c) A die is biased in such a way that the probability of throwing a 1 is $P(1) = \frac{x}{3}$ and the probability of throwing a 6 is $P(6) = \frac{1-x}{3}$. The probability of each of the other four numbers being thrown is equal.

(i) With a single toss, what is the probability of throwing a 4? **1**

(ii) If the die is tossed twice, write an expression for the probability of a total of 7. **2**

(iii) By using your result from (ii) find the value of x which will maximise the chance of a total of 7. **2**

Question 4 **Marks**

(a) (i) On the same set of axes sketch the curves $f(x)=x^2$ and $g(x)=\frac{1}{x}$ and hence $R(x)=f(x)+g(x)$ **3**

(ii) On separate sets of axes, sketch

(α) $y=|R(x)|$ **1**

(β) $y=R'(x)$ **2**

(γ) $y=[R(x)]^{-1}$ **2**

(b) An ellipse has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(i) Show that the equation of a tangent to the ellipse at the point (x_1, y_1) on the ellipse is $\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=1$ **3**

(ii) Two points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse such that $\theta+\phi=\frac{\pi}{2}$

(α) Find the coordinates of M the midpoint of P and Q in terms of θ **1**

(β) Hence show that the locus of M is a straight line passing through the origin. **1**

(c) On an Argand diagram, shade the region defined by $1\le|Z|\le 2$ and $\frac{\pi}{4}\le \arg Z\le\frac{\pi}{2}$ where Z is a complex number. **2**

Question 5 **Marks**

(a) A solid is formed such that its base is the parabola $y = x^2$ where $0 \le y \le a$. Cross- sections of the solid, perpendicular to the base are squares. By using the method of slicing, determine the volume of this solid. **3**

(b) A cylindrical hole of radius 1cm is bored through the centre of a sphere of radius 3cm. Calculate the exact volume of the sphere which remains. **4**

(c) A curve has parametric equations $x = \sin\theta$ and $y = \cos 2\theta$. Find the Cartesian equation of this curve. **1**

(d) A particle of weight 4 newtons, initially at rest, falls in a medium with retarding force kv newtons where k is a constant and v is the velocity (ms^{-1}) at t seconds after release.

(i) Show that an equation for the motion is $\ddot{x} = 1 - \dfrac{kv}{4}$ **1**

(ii) If the particle reaches a terminal velocity of 20 ms^{-1}, evaluate k. **1**

(iii) Show that the velocity of the particle at any time t seconds is $v = 20 - 20e^{\frac{-t}{20}}$. **3**

(iv) Determine an expression for the distance the particle has fallen through the medium after t seconds. **1**

Question 6 **Marks**

(a) (i) Use integration by parts to show that a reduction (recurrence) formula for $I_n = \int \sin^n x \; dx$ is $I_n = -\frac{1}{n}\cos x \; \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$ **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x dx$ **2**

(b) Verify the Pascal's Triangle relationship for Binomial Expansions **3**
i.e. Show that ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

(c) By considering the expansion of $(1+x)^n$, show that $\sum_{k=0}^{n} {}^nc_k = 2^n$ **2**

(d) A three digit number has a hundreds digit of a, a tens digit of b and a units digit of c. **2**
If a + b + c is divisible by 3, show that the three digit number is divisible by 3.

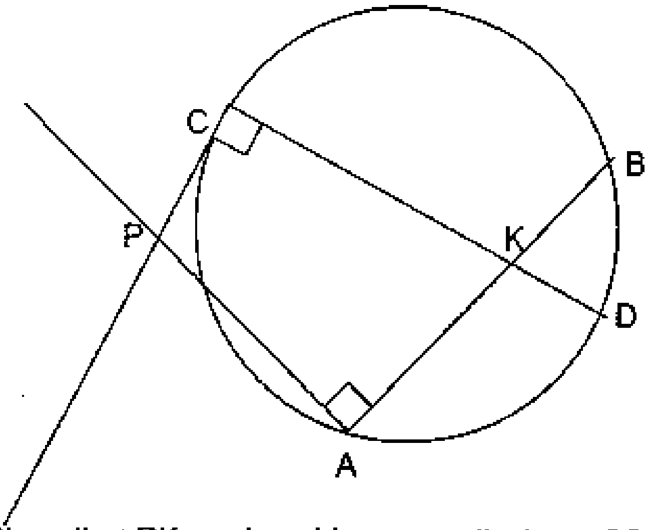
(e) Three identical yellow, four identical green and five identical blue beads are arranged on a circular bangle. How many arrangements will there be if:

(i) There are no restrictions on the placement of the beads ? **1**

(ii) All three yellow beads can not be together ? **2**

Question 7 **Marks**

(a) As shown below AB and CD are chords of a circle intersecting at K. P is a point such that $\angle DCP = \angle BAP = 90^0$



Show that PK produced is perpendicular to BD 4

(b) A light inextensible string of length 5 metres has one end fixed at a point A and the other fixed at B, 4 metres directly below A. A particle P of mass m is fixed to the midpoint of the string and moves in a horizontal circle which has its centre at the midpoint of AB, with velocity V ms^{-1}. The string parts AP and BP remain taut at all times.

(i) Find the tension in AP and in BP 4

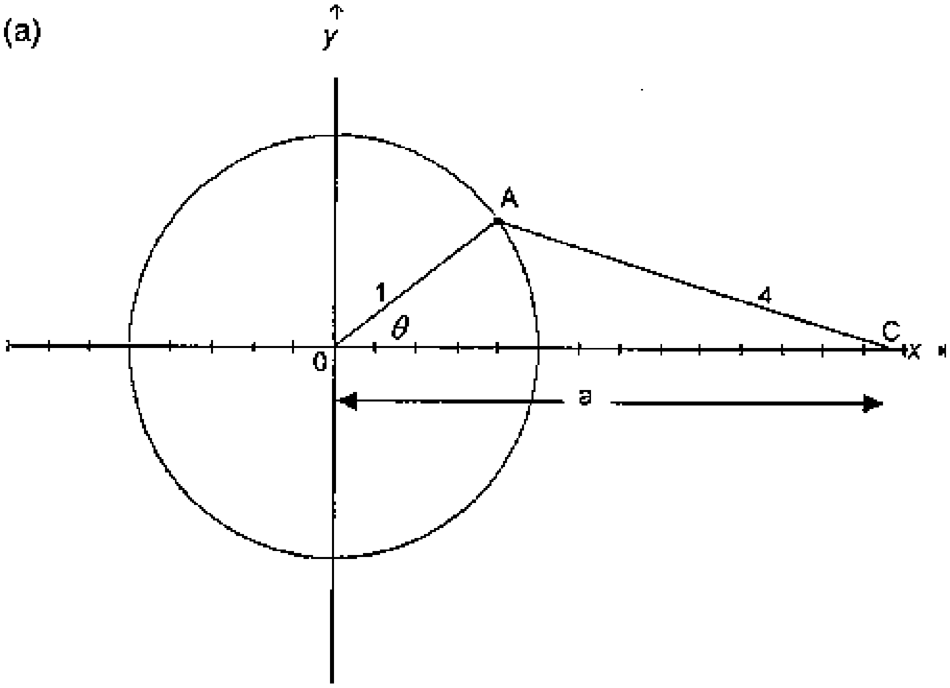
(ii) Show that for this motion to occur $8V^2 \geq 9g$ 2

(c) A corner on a race track is an arc of a circle of radius 100m. The track is banked such that there is no tendency for a vehicle to move sideways when cornering at 100km/h. Find the angle of banking. ($g = 10\text{ms}^{-2}$) 3

(d) Find $\dfrac{dy}{dx}$ for $x^2 + xy + y^2 = 1$

Question 8 **Marks**

(a)



As shown above the distances OA and OC are 1 metre and 4 metres respectively.
A moves anticlockwise on the circle whilst C can only slide along the X axis.
The distance of C from O at any time is a metres.

(i) Derive a relationship between θ and a. (Hint. Use the cosine rule). 1

(ii) If A is rotating anticlockwise at π radians per second, show that the velocity at which C is moving is given by $\dfrac{da}{dt} = \dfrac{-2a^2\pi\sin\theta}{a^2+15}$ 3

(iii) Find the velocity of C at the instant at which $\theta = \dfrac{\pi}{2}$ radians. 2

(b) If $u_1 = 5$, $u_2 = 11$ and $u_n = 4u_{n-1} - 3u_{n-2}$ for $n \geq 3$. Show by Mathematical Induction that $u_n = 2 + 3^n$ for $n = 1,2,3,4,...$ 5

(c) A geometric series has a general term $T_n = ar^{n-1}$. Write an expression for:

(i) The product of the first n terms 2

(ii) The sum of the reciprocals of the first n terms. 2

End of Paper