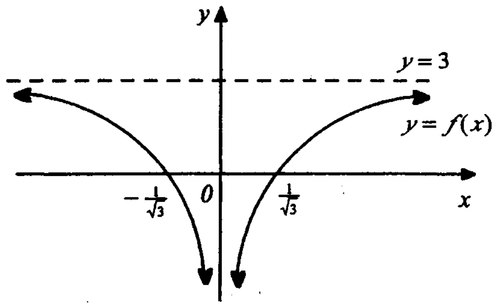
CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES
2005
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Question 1 **Marks**

(a) The diagram below shows the graph $y=f(x)$ where $f(x)=3-\dfrac{1}{x^2}$.



On separate diagrams, sketch the following graphs, in each case showing any intercepts on the coordinate axes and the equations of any asymptotes:

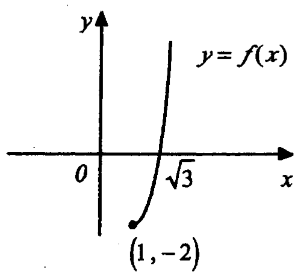
(i) $y=\{f(x)\}^2$ **2**

(ii) $y^2=f(x)$ **2**

(b)(i) The polynomial equation $P(x)=0$ has a double root α. Show that α is also a root of the equation $P'(x)=0$. **2**

(ii) The line $y=mx$ is a tangent to the curve $y=3-\dfrac{1}{x^2}$. Show that the equation $mx^3-3x^2+1=0$ has a double root and hence find any values of m. **4**

(c) The diagram below shows the graph $y=f(x)$ where $f(x)=x^3-3x,\ \ x\geq 1$.



(i) Copy the diagram. On your diagram sketch the graph of the inverse function $y=f^{-1}(x)$ showing any intercepts on the coordinate axes and the coordinates of any endpoints. Draw in the line $y=x$. **2**

(ii) Find the coordinates of any points of intersection of the curves $y=f(x)$ and $y=f^{-1}(x)$. Hence find the area of the region in the first quadrant bounded by the curves $y=f(x)$ and $y=f^{-1}(x)$ and the coordinate axes. **3**

Marks

Question 2 **(Begin a new page)**

(a) Find $\int \frac{1-\sin x}{\cos^2 x}\, dx$. **2**

(b) Find $\int \left(e^x + e^{-\frac{1}{2}x}\right)^2\, dx$. **2**

(c) Use the substitution $u=\sqrt{x}$ to evaluate $\int_1^{25} \frac{1}{x+\sqrt{x}}\, dx$, expressing the answer in simplest exact form. **3**

(d) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{5-4\cos x}\, dx$, expressing the answer in simplest exact form. **3**

(e)(i) If $I_n = \int_0^1 x(1-x)^n\, dx$, $n=0,1,2,\ldots$, show that $I_n = \frac{n}{n+2} I_{n-1}$, $n=1,2,3,\ldots$. **3**

(ii) Hence show that $I_n = \frac{1}{2\ {}^{n+2}C_2}$, $n=1,2,3,\ldots$ **2**

Question 3 **(Begin a new page)**

(a) Show that the complex number $z=\frac{6-2i}{3+4i} - \frac{6}{5i}$ is real. **2**

(b) $z_1 = 4\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ and $z_2 = 2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$.

(i) On an Argand diagram draw the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ representing z_1, z_2, z_1+z_2 respectively. **2**

(ii) Hence find $|z_1+z_2|$ in simplest exact form. **2**

(c) The quadratic equation $z^2+kz+4=0$, k real and $-4<k<4$, has two non-real roots α, β.

(i) Explain why α, β are complex conjugates. Hence show that $|\alpha|=|\beta|=2$. **2**

(ii) If α, β have arguments $\frac{\pi}{4}$, $-\frac{\pi}{4}$, find the value of k. **2**

(d)(i) On an Argand diagram shade the region where both $|z-(1+i)|\le\sqrt{2}$ and $0\le\arg z\le\frac{\pi}{2}$ **2**

(ii) Find the exact perimeter and the exact area of the shaded region. **3**

Marks

Question 4 **(Begin a new page)**

(a) Sketch the graph of the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ showing the intercepts on the axes, the coordinates of the foci and the equations of the directrices. **4**

(b) The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a>b>0$, has eccentricity e.

(i) Show that the line through the focus $F(ae,0)$ that is perpendicular to the asymptote $y=\frac{bx}{a}$ has equation $ax+by-a^2e=0$. **1**

(ii) Show that this line meets the asymptote at a point on the corresponding directrix. **3**

(c) $P\left(p,\frac{1}{p}\right)$ and $Q\left(q,\frac{1}{q}\right)$ are two variable points on the rectangular hyperbola $xy=1$ such that the chord PQ passes through the point $A(0,2)$. M is the midpoint of PQ.

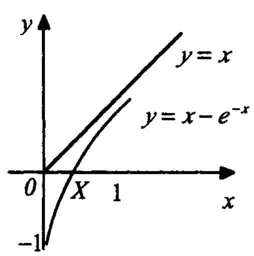
(i) Show that PQ has equation $x+pqy-(p+q)=0$. Hence deduce that $p+q=2pq$. **3**

(ii) Deduce that the tangent drawn from the point A to the rectangular hyperbola touches the curve at the point $(1,1)$. **1**

(iii) Sketch the rectangular hyperbola showing the points P, Q, A and M. Find the equation of the locus of M and state any restrictions on the domain of this locus. **3**

Question 5 **(Begin a new page)**

(a)



The diagram shows the graph of the curve $y=x-e^{-x}$, $x\geq 0$. This curve makes an intercept X on the x-axis, where $0<X<1$. The region bounded by the curve and the line $y=x$ between $x=0$ and $x=X$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by $V=2\pi\int_0^X xe^{-x}\,dx$. **3**

(ii) Hence show that $V=2\pi\left(1-X-X^2\right)$ **3**

Marks

(b) $z = \cos\theta + i\sin\theta$

(i) Express $1+z$ in modulus argument form. Hence show that $(1+z)^4 = 16\cos^4\frac{\theta}{2}\left(\cos 2\theta + i\sin 2\theta\right)$. **3**

(ii) Use the Binomial Theorem expansion of $(1+z)^4$ to show that $1+4\cos\theta+6\cos 2\theta+4\cos 3\theta+\cos 4\theta = 16\cos^4\frac{\theta}{2}\cos 2\theta$, and find a corresponding expression for $4\sin\theta+6\sin 2\theta+4\sin 3\theta+\sin 4\theta$. **3**

(iii) Hence show that $\dfrac{4\sin\theta+6\sin 2\theta+4\sin 3\theta+\sin 4\theta}{1+4\cos\theta+6\cos 2\theta+4\cos 3\theta+\cos 4\theta} = \tan 2\theta$, **3**

and $\dfrac{4\sin\theta+4\sin 3\theta+\sin 4\theta}{1+4\cos\theta+4\cos 3\theta+\cos 4\theta} = \tan 2\theta$.

Question 6 **(Begin a new page)**

(a) A particle of mass m kg is dropped from rest in a medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the velocity of the particle is $v\text{ ms}^{-1}$. After t seconds the particle has fallen x metres and has velocity $v\text{ ms}^{-1}$ and acceleration $a\text{ ms}^{-2}$. Take the acceleration due to gravity as 10 ms^{-2}.

(i) Draw a diagram showing the forces acting on the particle. Hence show that $a = \dfrac{100-v^2}{10}$. **2**

(ii) Show that $t = \frac{1}{2}\ln\left(\dfrac{10+v}{10-v}\right)$. **2**

(iii)Find expressions in terms of t for v and x. **3**

(iv) Show that the terminal velocity is 10 ms^{-1}. Hence find the exact time taken and the exact distance fallen by the particle in reaching a speed equal to 80% of its terminal velocity. **3**

(b) The equation $x^3+px+q=0$ (where p, q real) has roots α, β, γ.

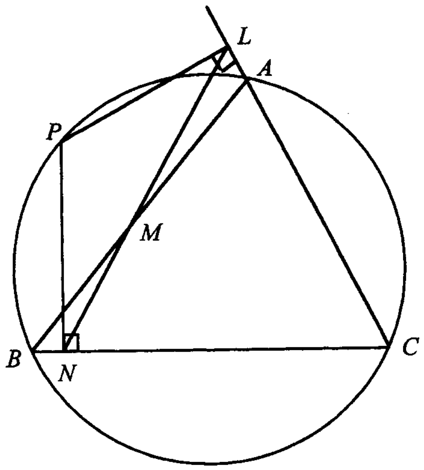
(i) Show that the monic cubic equation with roots α^2, β^2, γ^2 is $x^3+2px^2+p^2x-q^2=0$. **2**

(ii) Show that $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\gamma^2}{q}$. Hence find a cubic equation with roots $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$, $\dfrac{1}{\beta}+\dfrac{1}{\gamma}$ and $\dfrac{1}{\gamma}+\dfrac{1}{\alpha}$. **3**

Marks

Question 7 **(Begin a new page)**

(a)



ABC is an acute-angled triangle inscribed in a circle. P is a point on the minor arc AB of the circle. PL and PN are the perpendicuars from P to CA (produced) and CB respectively. LN cuts AB at M.

(i) Copy the diagram

(ii) Explain why $PNCL$ is a cyclic quadrilateral. **1**

(iii) Show that $\angle PBM = \angle PNM$. **3**

(iv) Hence show that PM is perpendicular to AB. **3**

(b) The equation $x^2 + x + 1 = 0$ has roots α, β. $T_n = \alpha^n + \beta^n$, $n = 1, 2, 3, \ldots$

(i) Show that $T_1 = T_2 = -1$. **2**

(ii) Show that $T_n = -T_{n-1} - T_{n-2}$, $n = 3, 4, 5, \ldots$ **2**

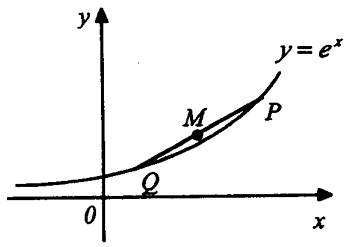
(iii)Hence use Mathematical Induction to show that $T_n = 2\cos\frac{2n\pi}{3}$, $n = 1, 2, 3, \ldots$ **4**

Marks

Question 8 **(Begin a new page)**

(a) A die is biased so that on any single roll the probability of getting an even score is p where $p \neq 0\cdot5$. In 12 rolls of this die the probability of getting exactly 4 even scores is three times the probability of getting exactly 3 even scores. Find the value of p. **3**

(b)



$P(a, e^a)$ and $Q(b, e^b)$, where $a > b$, are two points on the curve $y = e^x$.
M is the midpoint of PQ.

(i) Use the diagram to show that $e^a + e^b > 2e^{\frac{1}{2}(a+b)}$. **2**

(ii) Hence show that if $a > b > c > d$ then $e^a + e^b + e^c + e^d > 4e^{\frac{1}{4}(a+b+c+d)}$. **2**

(c) A closed hollow right cone with radius r and height h has volume V and surface area A.

(i) Show that $9V^2 = r^2A^2 - 2\pi r^4 A$. **3**

(ii) Hence show that if A is fixed then the maximum value of V is $\sqrt{\dfrac{A^3}{72\pi}}$. **5**