SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
HIGHER SCHOOL CERTIFICATE
TRIAL PAPER

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 120

- Attempt questions 1 – 8

Examiner: *C.Kourtesis*

NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate examination paper for this subject.

Section A
(Start a new answer sheet.)

Question 1. (15 marks)

Marks

(a) Evaluate $\int_0^2 \frac{3}{4+x^2}\,dx$. **2**

(b) Find $\int \cos x \sin^4 x \, dx$. **1**

(c) Use integration by parts to find **2**

$$\int te^{-t}dt .$$

(d) (i) Find real numbers a and b such that **2**

$$\frac{1}{x(\pi - 2x)} = \frac{a}{x} + \frac{b}{\pi - 2x} .$$

(ii) Hence find **2**

$$\int \frac{dx}{x(\pi - 2x)} .$$

(e) Evaluate $\int_{-3}^{3} (2 - |x|)dx$. **2**

(f) (i) Use the substitution $x = a - t$ to prove that **2**

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx .$$

(ii) Hence evaluate **2**

$$\int_0^{\frac{\pi}{2}} \log_e(\tan x)dx$$

Question 2. (15 marks)

Marks

(a) If $z = 2 + i$ and $w = -1 + 2i$ find **2**

$$\text{Im}(z - w).$$

(b) On an Argand diagram shade the region that is satisfied by both the conditions **2**

$$\text{Re}(z) \geq 2 \text{ and } |z - 1| \leq 2 .$$

(c) If $|z| = 2$ and $\arg z = \theta$ determine **3**

(i) $\left|\frac{i}{z^2}\right|$ (ii) $\arg\left(\frac{i}{z^2}\right)$

(d) If for a complex number z it is given that $\bar{z} = z$ where $z \neq 0$, determine the locus of z. **2**

(e) A complex number z is such that $\arg(z+2) = \frac{\pi}{6}$ and $\arg(z-2) = \frac{2\pi}{3}$. **3**

Find z, expressing your answer in the form $a + ib$ where *a* and *b* are real.

(f) The complex numbers z_1, z_2 and z_3 are represented in the complex plane by the points *P*, *Q* and *R* respectively. If the line segments *PQ* and *PR* have the same length and are perpendicular to one another, prove that: **3**

$$2z_1^2 + z_2^2 + z_3^2 = 2z_1(z_2 + z_3)$$

Section B
(Start a new answer sheet.)

Question 3. (15 marks)

Marks

(a) If $2-3i$ is a zero of the polynomial z^3+pz+q where p and q are real, find the values of p and q. **3**

(b) If α, β and γ are roots of the equation $x^3+6x+1=0$ find the polynomial equation whose roots are $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$. **2**

(c) Consider the function $f(x)=3\left(\frac{x+4}{x}\right)^2$.

(i) Show that the curve $y=f(x)$ has a minimum turning point at $x=-4$ and a point of inflexion at $x=-6$. **5**

(ii) Sketch the graph of $y=f(x)$ showing clearly the equations of any asymptotes. **2**

(d) Use mathematical induction to prove that **3**

$$n!>2^n \text{ for } n>3 \text{ where } n \text{ is an integer.}$$

Question 4 (15 marks)

(a) If $f(x) = \sin x$ for $-\pi \le x \le \pi$ draw neat sketches, on separate diagrams, of:

(i) $y = \left[f(x) \right]^2$ **2**

(ii) $y = \dfrac{1}{f\left(x + \dfrac{\pi}{2} \right)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = f\left(\sqrt{|x|} \right)$ **2**

(b) Show that the equation of the tangent to the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ at the point $P(x_0, y_0)$ on the curve is $xx_o^{-\frac{1}{2}} + yy_0^{-\frac{1}{2}} = a^{\frac{1}{2}}$. **3**

(c) Consider the polynomial $P(x) = x^5 - ax + 1$. By considering turning points on the curve $y = P(x)$, prove that $P(x) = 0$ has three distinct roots if **4**

$a > 5\left(\dfrac{1}{2} \right)^{\frac{8}{5}}$.

Section C
(Start a new answer booklet)

Question 5 (15 marks)

Marks

(a) A particle of mass m is thrown vertically upward from the origin with initial speed V_0. The particle is subject to a resistance equal to mkv, where v is its speed and k is a positive constant.

(i) Show that until the particle reaches its highest point the equation of motion is **1**

$$\ddot{y} = -(kv + g)$$

where y is its height and g is the acceleration due to gravity.

(ii) Prove that the particle reaches its greatest height in time T given by **4**

$$kT = \log_e\left[1 + \frac{kV_0}{g}\right].$$

(iv) If the highest point reached is at a height H above the ground prove that **4**

$$V_0 = Hk + gT\ .$$

(b) If α and β are roots of the equation $z^2 - 2z + 2 = 0$

(i) find α and β in mod-arg form. **3**

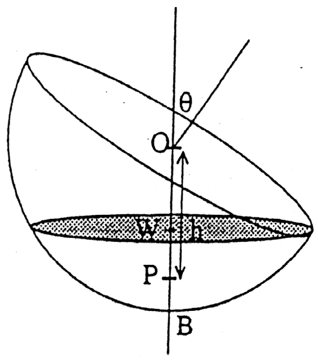
(ii) show that $\alpha^n + \beta^n = \sqrt{2^{n+2}}.\left[\cos\frac{n\pi}{4}\right].$ **3**

Question 6 (15 marks)

(a) A group of 20 people is to be seated at a long rectangular table, 10 on each side. There are 7 people who wish to sit on one side of the table and 6 people who wish to sit on the other side. How many seating arrangements are possible? **2**

(b) The area enclosed by the curves $y = \sqrt{x}$ and $y = x^2$ is rotated about the y axis through one complete revolution. Use the cylindrical shell method to find the volume of the solid that is generated. **3**

(c) The diagram shows a hemi-spherical bowl of radius r. The bowl has been tilted so that its axis is no longer vertical, but at an angle θ to the vertical. At this angle it can hold a volume V of water.



The vertical line from the centre O meets the surface of the water at W and meets the bottom of the bowl at B. Let P between W and B, and let h be the distance OP.

(i) Explain why $V = \int_{r\sin\theta}^{r} \pi\left(r^2 - h^2\right)dh$. **3**

(ii) Hence show $V = \dfrac{r^3\pi}{3}\left(2 - 3\sin\theta + \sin^3\theta\right)$. **2**

(d) (i) Show that $x^4 + y^4 \ge 2x^2y^2$. **2**

(ii) If $P(x, y)$ is any point on the curve $x^4 + y^4 = 1$ prove that $OP \le 2^{\frac{1}{4}}$, where O is the origin. **3**

Section D
(Start a new answer booklet)

Question 7 (15 marks)

(a) How many sets of 5 quartets (groups of four musicians) can be formed from 5 violinists, 5 viola players, 5 cellists, and 5 pianists if each quartet is to consist of one player of each instrument? **2**

(b) (i) If $t = \tan\theta$, prove that **2**

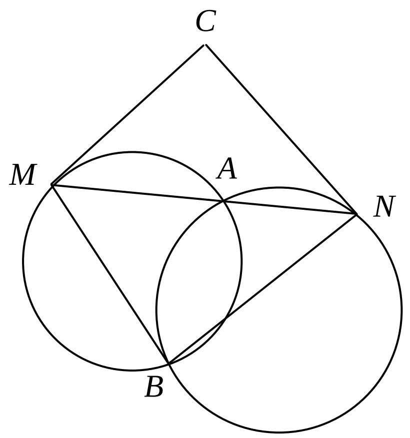
$$\tan 4\theta = \frac{4t\left(1-t^2\right)}{1-6t^2+t^4}.$$

(ii) If $\tan\theta\tan 4\theta = 1$ deduce that $5t^4 - 10t^2 + 1 = 0$. **2**

(iii) Given that $\theta = \frac{\pi}{10}$ and $\theta = \frac{3\pi}{10}$ are roots of the equation **4**

$\tan\theta\tan 4\theta = 1$, find the exact value of $\tan\frac{\pi}{10}$.

(c)



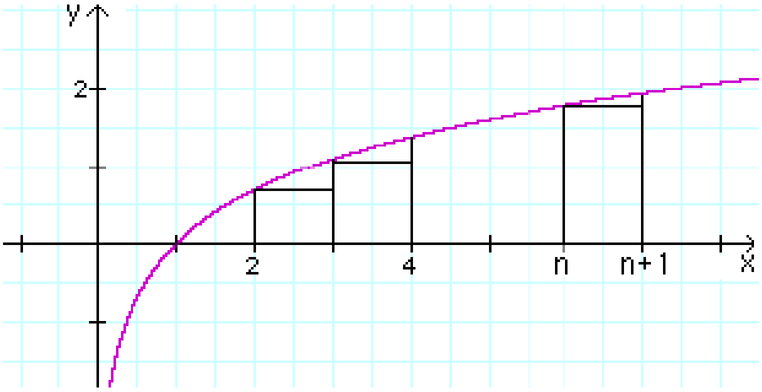
Two circles intersect at *A* and *B*. A line through *A* cuts the circles at *M* and *N*. The tangents at *M* and *N* intersect at *C*. **5**

(i) Prove that $\angle CMA + \angle CNA = \angle MBN$.

(ii) Prove *M*, *C*, *N*, *B* are concyclic.

Question 8 (15 marks)

(a) **6**



The diagram above shows the graph of $y = \log_e x$ for $1 \le x \le n+1$.

(i) By considering the sum of the areas of inner and outer rectangles show that

$$\ln(n!) < \int_1^{n+1} \ln x \, dx < \ln[(n+1)!]$$

(ii) Find $\int_1^{n+1} \ln x \, dx$.

(iii) Hence prove that

$$e^n > \frac{(n+1)^n}{n!}$$

(b) If a root of the cubic equation $x^3 + bx^2 + cx + d = 0$ is equal to the reciprocal of another root, prove that **3**

$$1 + bd = c + d^2.$$

This question continues on the next page.

(c) A stone is projected from a point O on a horizontal plane at an angle of elevation α and with initial velocity U metres per second. The stone reaches a point A in its trajectory, and at that instant it is moving in a direction perpendicular to the angle of projection with speed V metres per second. **6**

Air resistance is neglected throughout the motion and g is the acceleration due to gravity.

If t is the time in seconds at any instant, show that when the stone is at A:

(i) $V = U\cot\alpha$

(ii) $t = \dfrac{U}{g\sin\alpha}$.

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right)\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$