<div style="border:1px solid black">

2000 HIGHER SCHOOL CERTIFICATE
SOLUTIONS
3 UNIT (ADDITIONAL) AND
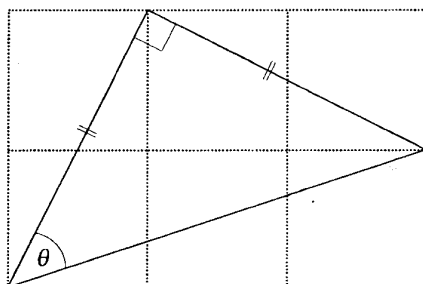# 3/4 UNIT (COMMON) MATHEMATICS

</div>

QUESTION 1

(a) $\dfrac{d}{dx} x\sin^{-1} x = \sin^{-1} x + \dfrac{x}{\sqrt{1-x^2}}$.

(b) $m_1 = 2, m_2 = \dfrac{1}{3}$.

$$\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \dfrac{2 - \dfrac{1}{3}}{1 + 2 \times \dfrac{1}{3}}$$

$$= 1.$$

$\therefore$ The acute angle is $45°$ $\left(\dfrac{\pi}{4} \text{ radians} \right)$.

Alternative solution:

The sketch shows two lines with gradients 2 and $\frac{1}{3}$. By considering the triangle formed (right-angled isosceles), the angle between the two lines, θ, is $45°$.

(c) Method 1:
By the factor theorem, P(3) = 0 when $x - 3$ is a factor.
$$P(3) = 27 - 9k + 6 = 0$$
$$9k = 33$$
$$k = 3\tfrac{2}{3}.$$

Method 2:
$$P(x) = (x-3)\left(x^2 + 3x - 2\right)$$
$$= x^3 - 11x + 6.$$
Equating coefficients, $k = \dfrac{11}{3} = 3\tfrac{2}{3}$.

Method 3:
$$P(x) = x^3 - 3kx + 6$$
Let roots be $3, \alpha, \beta$.

$$\alpha + \beta + 3 = 0 \qquad \text{①}$$
$$3\alpha + 3\beta + \alpha\beta = -3k \qquad \text{②}$$
$$3\alpha\beta = -6 \qquad \text{③}$$

From ①: $\alpha + \beta = -3$
From ③: $\alpha\beta = -2$
In ②: $3(\alpha + \beta) + \alpha\beta = -3k$
$$3(-3) - 2 = -3k$$
$$k = \dfrac{11}{3}.$$

(d) $\displaystyle\int_0^{\sqrt{3}} \dfrac{4}{x^2 + 9}\, dx = \left[\dfrac{4}{3} \tan^{-1} \dfrac{x}{3} \right]_0^{\sqrt{3}}$ (standard integral)

$$= \dfrac{4}{3}\left(\tan^{-1} \dfrac{\sqrt{3}}{3} - \tan^{-1} 0 \right)$$

$$= \dfrac{4}{3}\left(\dfrac{\pi}{6} - 0 \right)$$

$$= \dfrac{2\pi}{9}.$$

(e) Method 1:
$$\dfrac{5}{x+2} \le 1, \quad x \ne -2$$
$$\therefore \qquad 5(x+2) \le (x+2)^2$$
$$(x+2)^2 - 5(x+2) \ge 0$$
$$\therefore \ (x+2)(x+2-5) \ge 0$$
$$(x+2)(x-3) \ge 0.$$
$$\therefore \ x < -2 \text{ or } x \ge 3.$$

Method 2:
$$\dfrac{5}{x+2} \le 1, \quad x \ne -2$$
Consider $\dfrac{5}{x+2} = 1$
$$\therefore \qquad 5 = x + 2$$
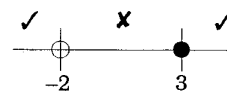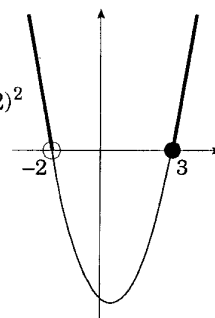$$x = 3.$$
$$\therefore \ x < -2 \text{ or } x \ge 3.$$

Method 3:
When $x + 2 > 0$, that is, $x > -2$,
$$5 \le x + 2,$$
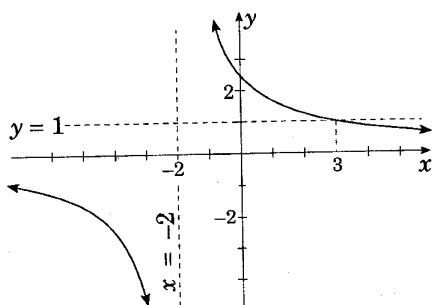that is, $x \ge 3$.

When $x + 2 < 0$, that is, $x < -2$,
$$5 \ge x + 2,$$

that is, $x \le 3$. Hence $x < -2$.

∴ Solution is $x < -2$ or $x \ge 3$.

Method 4:

Graph $y = \dfrac{5}{x+2}$ and $y = 1$.

Point of intersection when $x = 3$
(see Method 2).

From the graph, $x < -2$ or $x \ge 3$.

QUESTION 2

(a) 'O' is repeated 3 times.
  The other letters are unique.

∴ The number of arrangements $= \dfrac{9!}{3!}$

$= 60\,480$.

(b) $\left(5 + 2x^2\right)^7 = \displaystyle\sum_{r=0}^{7} {}^7C_r 5^{7-r}\left(2x^2\right)^r$.

For a term in x^6 we require $2r = 6$, ∴ $r = 3$.

∴ The coefficient of x^6 is ${}^7C_3\, 5^{7-3}\, 2^3 = 175\,000$.

(c) $\cos 2\theta = \sin\theta$,

∴

$1 - 2\sin^2\theta = \sin\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \dfrac{1}{2}$ or -1.

∴ $\theta = \dfrac{\pi}{6},\ \dfrac{5\pi}{6},\ \dfrac{3\pi}{2}$.

(d) $u = 2 + x,\ x = u - 2$, ∴ $\dfrac{dx}{du} = 1$.

$\displaystyle\int \dfrac{x}{\sqrt{2+x}}\, dx = \int \dfrac{u-2}{u^{\frac{1}{2}}}\, du$

$= \displaystyle\int \left(u^{\frac{1}{2}} - 2u^{-\frac{1}{2}} \right) du$

$= \dfrac{2u^{\frac{3}{2}}}{3} - 4u^{\frac{1}{2}} + C$

$= \dfrac{2}{3}(2+x)^{\frac{3}{2}} - 4\sqrt{2+x} + C$

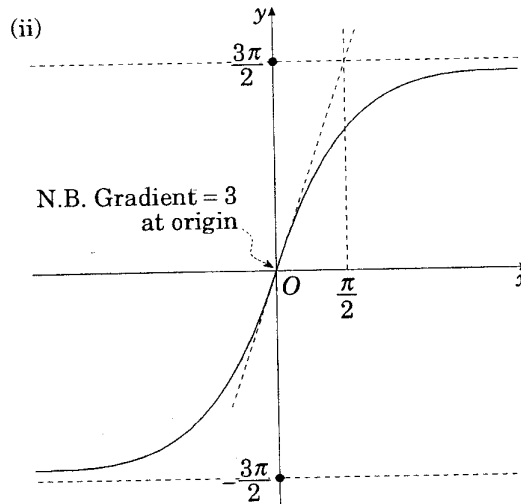$= \dfrac{2}{3}\sqrt{2+x}\left(2 + x - \dfrac{3}{2} \times 4 \right) + C$

$= \dfrac{2}{3}\sqrt{2+x}\,(x - 4) + C$.

QUESTION 3

(a) $f'(a) = \displaystyle\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{(a+h)^3 - a^3}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{h\left(3a^2 + 3ah + h^2\right)}{h}$

$= \displaystyle\lim_{h \to 0} \left(3a^2 + 3ah + h^2\right)$

$= 3a^2$.

(b) $f(x) = 3\tan^{-1}x$.

(i) Range is $-\dfrac{3\pi}{2} \le y \le \dfrac{3\pi}{2}$.

(ii)

N.B. Gradient = 3 at origin

(iii) $f'(x) = \dfrac{3}{1+x^2}$

$f'\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{3}{1 + \frac{1}{3}} = 2\dfrac{1}{4}$.

Therefore the gradient of the tangent at $x = \dfrac{1}{\sqrt{3}}$ is $2\dfrac{1}{4}$.

(c) (i) From $\triangle OTB$, $\dfrac{h}{OB} = \tan 30° = \dfrac{1}{\sqrt{3}}$,

∴ $OB = \sqrt{3}h$.

(Other answers are possible,
such as $OB = \sqrt{100^2 + h^2}$.)

(ii) From $\triangle OAT$, $\dfrac{OA}{h} = \tan 45° = 1$,

∴ $OA = h$.

From $\triangle OAB$, $OB^2 = AO^2 + AB^2$

$3h^2 = h^2 + 100^2$

$2h^2 = 100^2$

∴ $h = 50\sqrt{2}$.

(iii)

$$\sin\theta = \frac{OA}{OB}$$

$$= \frac{50\sqrt{2}}{50\sqrt{6}}$$

$$= \frac{1}{\sqrt{3}}.$$
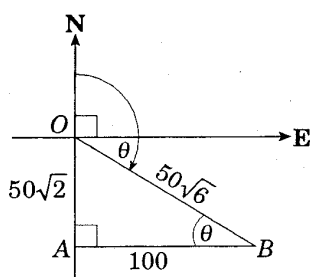
$$\therefore \quad \theta = 35°\,16'.$$

Therefore the bearing is $125°\,16'$.

QUESTION 4

(a) Let $S(n)$ be the statement

$$1+3+6+\cdots+\tfrac{1}{2}n(n+1) = \tfrac{1}{6}n(n+1)(n+2).$$

When $n = 1$,   LHS $= 1$

RHS $= \tfrac{1}{6}\times 1\times 2\times 3 = 1.$

$\therefore$ LHS $=$ RHS.

$\therefore$ $S(1)$ is true.

Assume $S(k)$ is true. That is,

$$1+3+6+\cdots+\tfrac{1}{2}k(k+1) = \tfrac{1}{6}k(k+1)(k+2).$$

Now

$$1+3+6+\cdots+\tfrac{1}{2}k(k+1)+\tfrac{1}{2}(k+1)(k+1+1)$$

$$= \tfrac{1}{6}k(k+1)(k+2)+\tfrac{1}{2}(k+1)(k+1+1)$$

$$\text{(by assumption)}$$

$$= \frac{(k+1)(k+2)}{6}(k+3)$$

$$= \tfrac{1}{6}(k+1)(k+1+1)(k+1+2).$$

This is $S(k+1)$.

$\therefore$ If $S(k)$ is true, then $S(k+1)$ is true.

But $S(1)$ is true, hence $S(2)$ is true, hence $S(3)$ is true and so on.

By the principle of mathematical induction, the result is true for all positive integral values of n.

(b) Let $f(r) = (1+r)\!\left[(1+r)^{24}-1\right]-50r$

$$= (1+r)^{25}-(1+r)-50r$$

$$= (1+r)^{25}-1-51r.$$

$$f'(r) = 25(1+r)^{24}-51$$

$$r_1 = 0{\cdot}06,$$

then $r_2 = r_1 - \dfrac{f(r_1)}{f'(r_1)}$

$$= 0{\cdot}06 - \frac{(1{\cdot}06)^{25}-1-51\times 0{\cdot}06}{25(1{\cdot}06)^{24}-51}$$

$$= 0{\cdot}055\,38\ldots$$

$$= 0{\cdot}055.$$

(c) $P(x) = x^3 + px^2 + qx + r$.

(i) Sum of roots $= -\dfrac{b}{a}$.

$$\therefore \quad \sqrt{k}+\left(-\sqrt{k}\right)+\alpha = -p$$

$$\therefore \quad\quad\quad \alpha + p = 0.$$

(ii) Product of roots $= -\dfrac{d}{a}$.

$$\therefore \quad \sqrt{k}\times\left(-\sqrt{k}\right)\times\alpha = -r$$

$$\therefore \quad\quad\quad -k\alpha = -r$$

$$k\alpha = r.$$

(iii) Product of roots in pairs $= \dfrac{c}{a}$.

$$\therefore \quad \sqrt{k}\times\left(-\sqrt{k}\right)+\sqrt{k}\times\alpha+\left(-\sqrt{k}\right)\times\alpha = q$$

$$\therefore \quad\quad\quad -k = q.$$

Since $\alpha = -p$, from (i), then $k\alpha = r$ from (ii) becomes $(-q)\times(-p) = r$, that is, $pq = r$.

(d) Period $T = \dfrac{2\pi}{n} = 4\pi$,

$$\therefore \quad n = \tfrac{1}{2}.$$

Amplitude $a = 3$.

Since $v^2 = n^2\!\left(a^2-x^2\right)$ (putting origin at O)

$$v^2 = \tfrac{1}{4}\!\left(9-x^2\right).$$

When $x = 0$, at O, $v^2 = \dfrac{9}{4}$.

$$\therefore \quad \text{Speed} = \dfrac{3}{2}\ \text{cm s}^{-1}.$$

QUESTION 5

(a)
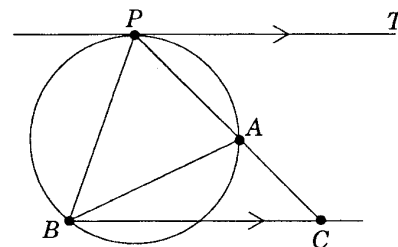
(i) $\angle PBA = \angle TPA$ ($\angle$ between tangent and chord equals $\angle$ in alternate segment).

$\angle TPA = \angle PCB$ (alternate $\angle$s, $PT\|BC$),

$\therefore$ $\angle PBA = \angle PCB$.

(ii) In $\triangle PBA$ and $\triangle PCB$,

$\angle APB = \angle BPC$ (common)

$\angle PBA = \angle PCB$, from (i).

Therefore the triangles are equiangular, and hence similar.

$\therefore \quad \dfrac{PB}{PC} = \dfrac{PA}{PB}$ (corresponding sides are in the same ratio)

$\therefore \quad PB^2 = PA \times PC.$

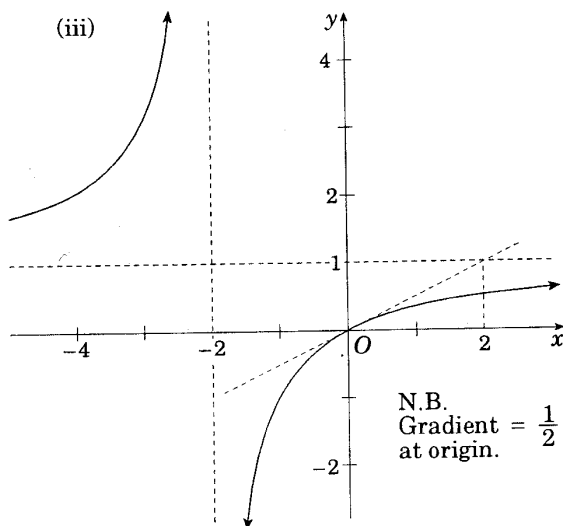**(b)** $f(x) = \dfrac{x}{x+2}$, defined for all $x \neq -2$.

(i) $f'(x) = \dfrac{(x+2) \times 1 - x \times 1}{(x+2)^2}$

$\qquad = \dfrac{2}{(x+2)^2} > 0$ for all $x \neq -2$,

since $(x+2)^2 > 0$ for all $x \neq -2$.

(ii) $f(x) = \dfrac{x+2-2}{x+2}$

$\qquad = 1 - \dfrac{2}{x+2}.$

As $x \to \pm\infty$, $\dfrac{2}{x+2} \to 0$.

Therefore the horizontal asymptote is $y = 1$.

(iii)

N.B.
Gradient $= \dfrac{1}{2}$
at origin.

(iv) $f(x)$ is a one-to-one increasing function (it satisfies the horizontal-line test).

(v) $y = \dfrac{x}{x+2}.$

$\therefore$ The inverse is $x = \dfrac{y}{y+2}.$

$\therefore \qquad xy + 2x = y$

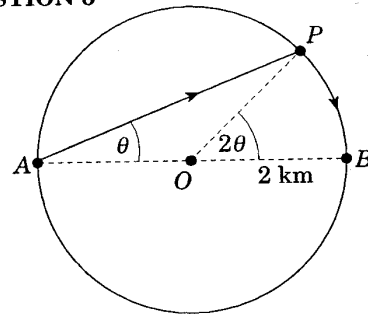$\qquad\qquad 2x = y(1-x)$

$\qquad\qquad\ y = \dfrac{2x}{1-x}$

$\therefore \qquad f^{-1}(x) = \dfrac{2x}{1-x}.$

(vi) Domain of $f^{-1}(x)$ is the range of $f(x)$.
That is, all real x, $x \neq 1$.

QUESTION 6

(a)

(i) $\angle APB = \dfrac{\pi}{2}$ ($\angle$ in a semicircle).

$\therefore \dfrac{AP}{AB} = \cos\theta$

$\therefore \quad AP = 4\cos\theta.$

Arc $PB = 2 \cdot 2\theta = 4\theta.$

Time from A to B
= time for AP + time for PB.

That is, $\quad T = \dfrac{AP}{3} + \dfrac{PB}{4}$

$\qquad\qquad = \dfrac{4\cos\theta}{3} + \theta$

$\qquad\qquad = \dfrac{1}{3}(4\cos\theta + 3\theta).$

(ii) $\dfrac{dT}{d\theta} = \dfrac{1}{3}(-4\sin\theta + 3)$

$\qquad = 0$ when $\sin\theta = \dfrac{3}{4}.$

$\therefore \qquad \theta = \sin^{-1}\left(\dfrac{3}{4}\right)$

$\qquad\qquad = 0.848$ radians $(= 48°\,35').$

(iii) $\dfrac{d^2T}{d\theta^2} = \dfrac{1}{3}(-4\cos\theta) < 0$ for θ acute.

Therefore this is a maximum stationary point, and so not the minimum.

Test the end points.
That is, row direct to B.

Time $= \dfrac{4}{3} = 1\dfrac{1}{3}$ hours.

Walk round the lake from A to B.

Time $= \dfrac{2\pi}{4} = 1.57$ hours.

$\therefore$ Pat should row directly across the lake to B to minimise the time.

(b) (i) No. of ways for 2 spades and 4 clubs
$= {}^{13}C_2 \times {}^{13}C_4 = 55\,770.$

(ii) No. of ways with 5 of the same suit
$= 4 \times {}^{13}C_5 \times {}^{39}C_1 = 200\,772.$

No. of ways with 6 of the same suit
$= 4 \times {}^{13}C_6 = 6864.$

$\therefore$ Total number with at least 5 of the same suit $= 200\,772 + 6864 = 207\,636.$

QUESTION 7

(a) (i) $F = Au^3 + \dfrac{B}{u}$.

The maximum period of flight implies the minimum fuel used per hour.

$$\frac{dF}{du} = 3Au^2 - \frac{B}{u^2}.$$

$$\frac{d^2F}{du^2} = 6Au + \frac{2B}{u^3} > 0,$$

since $u > 0$ and A and B are positive. Therefore we have a minimum when

$$\frac{dF}{du} = 0,$$

that is, when $3Au^2 = \dfrac{B}{u^2}$

$$u^4 = \frac{B}{3A}$$

$$u = \left(\frac{B}{3A}\right)^{\frac{1}{4}}.$$

(ii) Let the distance be s and the time t.

$s = ut$

$\quad = u\dfrac{k}{F}$ (where k is a positive constant, the amount of fuel)

$\quad = \dfrac{uk}{Au^3 + \dfrac{B}{u}}$

$\quad = \dfrac{u^2k}{Au^4 + B}.$

$$\frac{ds}{du} = \frac{2uk(Au^4 + B) - u^2k \cdot 4Au^3}{(Au^4 + B)^2}$$

$$= \frac{2Aku^5 + 2Bku - 4Aku^5}{(Au^4 + B)^2}$$

$$= \frac{2Bku - 2Aku^5}{(Au^4 + B)^2}$$

$$= \frac{2ku(B - Au^4)}{(Au^4 + B)^2}.$$

For a maximum s, $\dfrac{ds}{du} = 0.$

$\therefore\ u = 0$ or $u^4 = \dfrac{B}{A}.$

When $u = 0$, s is obviously a minimum value,

$\therefore$ maximum occurs when $u = \left(\dfrac{B}{A}\right)^{\frac{1}{4}}.$

$$\frac{\text{New speed}}{\text{Old speed}} = \frac{\left(\dfrac{B}{A}\right)^{\frac{1}{4}}}{\left(\dfrac{B}{3A}\right)^{\frac{1}{4}}}$$

$$= 3^{\frac{1}{4}}$$

$$= 1 \cdot 3162$$

$$= 132\% \text{ (nearest per cent)}.$$

Therefore the speed for maximum distance is approximately 32% faster than the speed for maximum time.

(b) (i) Given $x = Vt\cos\theta$

$\therefore\ t = \dfrac{x}{V\cos\theta}$

$$y = Vt\sin\theta - \frac{1}{2}gt^2$$

$$= \frac{V \cdot x\sin\theta}{V\cos\theta} - \frac{1}{2}g\frac{x^2}{V^2\cos^2\theta}$$

$$= x\tan\theta - x^2\sec^2\theta\left(\frac{g}{2V^2}\right)$$

$$= x\tan\theta - x^2\sec^2\theta, \text{ since } \frac{2V^2}{g} = 1.$$

(ii) On the inclined plane,

$$x = r\cos\alpha, \ y = r\sin\alpha.$$

$$\therefore\ r\sin\alpha = r\cos\alpha\tan\theta - \frac{r^2\cos^2\alpha}{\cos^2\theta}.$$

$\sin\alpha\cos^2\theta$

$= \cos\alpha\sin\theta\cos\theta - r\cos^2\alpha$ (since $r \neq 0$)

$\therefore\ r\cos^2\alpha = \cos\theta(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$

$$\therefore\ \quad r = \frac{\cos\theta \cdot \sin(\theta - \alpha)}{\cos^2\alpha}.$$

(iii) Using the given identity with $A = \theta - \alpha$ and $B = \theta$, we have

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha}$$

$$= \frac{\sin(2\theta - \alpha) + \sin(-\alpha)}{2\cos^2\alpha}.$$

This will be a maximum when $\sin(2\theta - \alpha) = 1.$

$$\left(\text{That is, when } 2\theta - \alpha = \frac{\pi}{2}.\right)$$

So the maximum range is

$$R = \frac{1 + \sin(-\alpha)}{2\cos^2\alpha}$$

$$= \frac{1 - \sin\alpha}{2\left(1 - \sin^2\alpha\right)}$$

$$= \frac{1 - \sin\alpha}{2(1 - \sin\alpha)(1 + \sin\alpha)}$$

$$= \frac{1}{2(1 + \sin\alpha)}.$$

(iv) **Method 1**:

Let m_O and m_T be the slopes of the tangents at O and T respectively.

$$y = x\tan\theta - x^2\sec^2\theta$$

$$y' = \tan\theta - 2x\sec^2\theta.$$

At O, $x = 0$, so $m_O = \tan\theta.$

Equation of OT is $y = x\tan\alpha$,

so T satisfies $x\tan\alpha = x\tan\theta - x^2\sec^2\theta.$

$\therefore\ x\sec^2\theta = \tan\theta - \tan\alpha$ $(x \neq 0$ at $T)$

$$x = \frac{\tan\theta - \tan\alpha}{\sec^2\theta}.$$

$$\therefore \quad m_T = \tan\theta - 2\left(\frac{\tan\theta - \tan\alpha}{\sec^2\theta}\right)\cdot\sec^2\theta$$

$$= 2\tan\alpha - \tan\theta$$

$$= 2\tan\left(2\theta - \frac{\pi}{2}\right) - \tan\theta \quad \text{(from (iii))}$$

$$= -2\cot 2\theta - \tan\theta$$

$$= -2\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - \tan\theta$$

$$= -\frac{1}{\tan\theta} + \tan\theta - \tan\theta$$

$$= -\frac{1}{\tan\theta}.$$

$$m_O m_T = \tan\theta \times\left(-\frac{1}{\tan\theta}\right)$$

$$= -1.$$

Therefore the tangents are perpendicular.

Method 2:
The endpoints of a focal chord have tangents which are perpendicular, so it is enough to show that for maximum range OT is a focal chord.
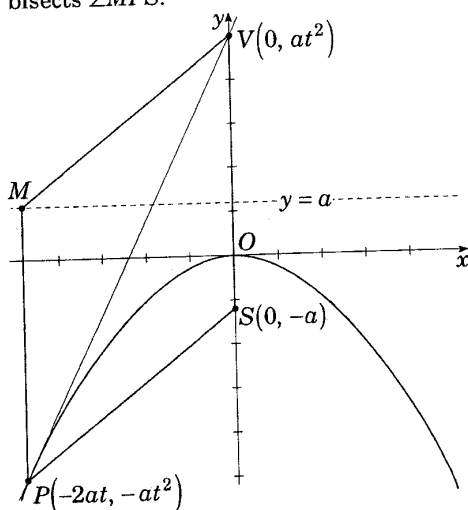
Method 2a (focal chord from geometry):
The following diagram shows a general parabola $x^2 = -4ay$ with focus $S(0, -a)$.
The tangent at $P\left(y = tx + at^2\right)$ intersects the y axis at $V\left(0, at^2\right)$.

Therefore $PM = SV\left(= at^2 + a\right)$.

Also, $PM = PS$ by definition of a parabola. Therefore $PMVS$ is a rhombus, and PV bisects $\angle MPS$.

This only works because PS is a focal chord. For any other point S on the y axis, $\angle VPS$ would be different.
So if a tangent bisects the angle between the vertical and a chord, that chord must be a focal chord.

In part (iii) we showed that the maximum range occurs when
$$2\theta - \alpha = \frac{\pi}{2}.$$
$$\therefore \quad \theta - \alpha = \frac{\pi}{2} - \theta.$$
This means that the tangent at O (with angle θ) bisects the vertical (with angle $\frac{\pi}{2}$) and OT (with angle α).
Hence OT is a focal chord as required.

Method 2b (focal chord using algebra):
$$y = x\tan\theta - x^2\sec^2\theta$$
$$\therefore \quad \cos^2\theta\cdot y = x\sin\theta\cos\theta - x^2.$$
$$\left(x - \frac{\sin\theta\cos\theta}{2}\right)^2 = \frac{\sin^2\theta\cos^2\theta}{4} - \cos^2\theta\cdot y$$
$$= -\cos^2\theta\left(y - \frac{\sin^2\theta}{4}\right).$$

Focal length $= \dfrac{\cos^2\theta}{4}$.

Vertex $= \left(\dfrac{\sin\theta\cos\theta}{2}, \dfrac{\sin^2\theta}{4}\right)$.

Focus $F = \left(\dfrac{\sin\theta\cos\theta}{2}, \dfrac{\sin^2\theta - \cos^2\theta}{4}\right)$.

Slope of $OF = \dfrac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta}$
$$= \frac{1}{2}\left(\tan\theta - \frac{1}{\tan\theta}\right).$$

Now for maximum range, $2\theta - \alpha = \frac{\pi}{2}$.
$$\tan 2\theta = \tan\left(\alpha + \frac{\pi}{2}\right)$$
$$= -\frac{1}{\tan\alpha}$$
$$\therefore \quad \tan\alpha = -\frac{1}{\tan 2\theta}$$
$$= -\frac{1-\tan^2\theta}{2\tan\theta}$$
$$= \frac{1}{2}\left(\tan\theta - \frac{1}{\tan\theta}\right).$$

Therefore the slope of OF equals the slope of OT, and so OT is a focal chord, as required.

END OF 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS SOLUTIONS