Student Number___________________

ABBOTSLEIGH

AUGUST 2008
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

Outcomes assessed

HSC course

- **E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- **E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- **E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- **E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- **E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- **E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- **E7** uses the techniques of slicing and cylindrical shells to determine volumes
- **E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- **E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

From the Extension 1 Mathematics Course

Preliminary course

- **PE1** appreciates the role of mathematics in the solution of practical problems
- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives that require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks)
Use a SEPARATE writing booklet.

(a) Using the table of standard integrals find $\int \frac{dx}{\sqrt{x^2+7}}$ **1**

(b) By completing the square find $\int \frac{dx}{\sqrt{4x-x^2}}$ **2**

(c) Find $\int \frac{1-2x}{\sqrt{1-x^2}}dx$, $|x|<1$. **2**

(d) Find $\int \cos^3 x\, dx$. **2**

(e) (i) Use the substitution $x=\frac{2}{3}\sin\theta$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\, dx = \frac{\pi}{3}$. **3**

(ii) Hence or otherwise, find the area enclosed by the ellipse $\frac{9x^2}{4}+\frac{y^2}{4}=1$. **2**

(f) Evaluate $\int_0^1 \tan^{-1} x\, dx$. **3**

Marks

QUESTION 2 (15 marks)
Start a new writing booklet.

(a) Given $z_1 = 3 - i$ and $z_2 = 2 + 5i$, express the following in the form $a + ib$ where a and b are real:

(i) $(\overline{z}_1)^2$ **2**

(ii) $\dfrac{z_1}{z_2}$ **2**

(iii) $|z_1 z_2|$ **2**

(b) (i) Sketch the region $|z + 1 + i| \leq 1$. **2**

(ii) Find the maximum and minimum values of $|z|$. **2**

(c) (i) The complex number $z = x + iy$ is represented by the point P. If $\dfrac{z-1}{z-2i}$ is purely imaginary, show that the locus of P is the circle $x^2 - x + y^2 - 2y = 0$. **3**

(ii) Sketch this locus showing all important features. **2**

Marks

QUESTION 3 (15 marks)
Start a new writing booklet.

(a) Sketch on separate diagrams, the graphs of:

(i) $y = (x-1)^2(x+2)$ **1**

(ii) $y^2 = (x-1)^2(x+2)$ **2**

(iii) $y = \dfrac{1}{(x-1)^2(x+2)}$ **2**

(b) Sketch $y = \log_e (x+1)^2$ **2**

(c) Sketch the graph of the function $y = \dfrac{x^2 - x + 1}{(x-1)^2}$, clearly showing the coordinates of any points of intersection with the x and y axes, the coordinates of any turning points and the equations of any asymptotes. There is no need to investigate points of inflexion. **4**

(d) If α, β and γ are the roots of $x^3 + 2x^2 - 3x - 4 = 0$,

(i) Evaluate $\alpha^2 + \beta^2 + \gamma^2$. **2**

(ii) Form the equation whose roots are $\beta\gamma$, $\alpha\gamma$ and $\alpha\beta$. **2**

Marks

QUESTION 4 (15 marks)
Start a new writing booklet.

(a) The foci of a hyperbola of eccentricity $\frac{13}{12}$ are the points $(\pm 13, 0)$.

(i) Show that the equation of the hyperbola is $\frac{x^2}{144} - \frac{y^2}{25} = 1$. **2**

(ii) Find the equation of the tangent to the hyperbola at the point $(12\sec\theta, 5\tan\theta)$. **3**

(b) (i) Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$. **2**

(ii) Show that the pair of tangents drawn from the point (3, 4) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to each other. **3**

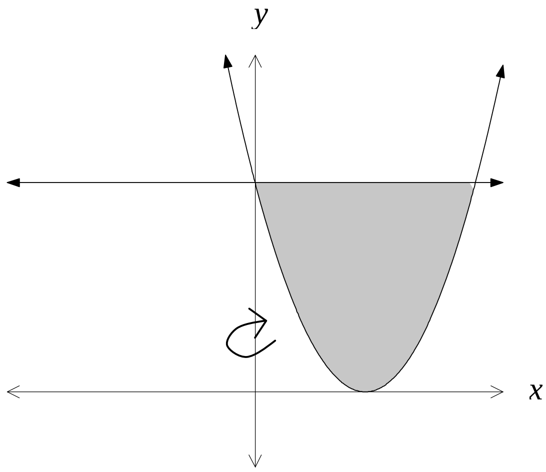
(c) (i) Verify that $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a root of $z^5 + z - 1 = 0$. **2**

(ii) Find the monic cubic equation with real coefficients whose roots are also the roots of $z^5 + z - 1 = 0$ but do not include α. **3**

QUESTION 5 (15 marks)
Start a new writing booklet.

(a) The base of a certain solid is a circle with radius 2. Each parallel cross-section of the solid is a square. Find the volume of the solid. **3**

(b) The area enclosed by the curve $y=(x-2)^2$ and the line $y=4$ is rotated around the y – axis. Use the method of cylindrical shells to find the volume formed. **3**



(c) (i) Show that the tangent to the rectangular hyperbola $xy=4$ at the point $T\left(2t, \dfrac{2}{t}\right)$ has equation $x+t^2y=4t$. **2**

(ii) This tangent cuts the x-axis at the point Q. Find the coordinates of Q. **1**

(iii) Show that the line through Q which is perpendicular to the tangent at T has equation $t^2x-y=4t^3$. **1**

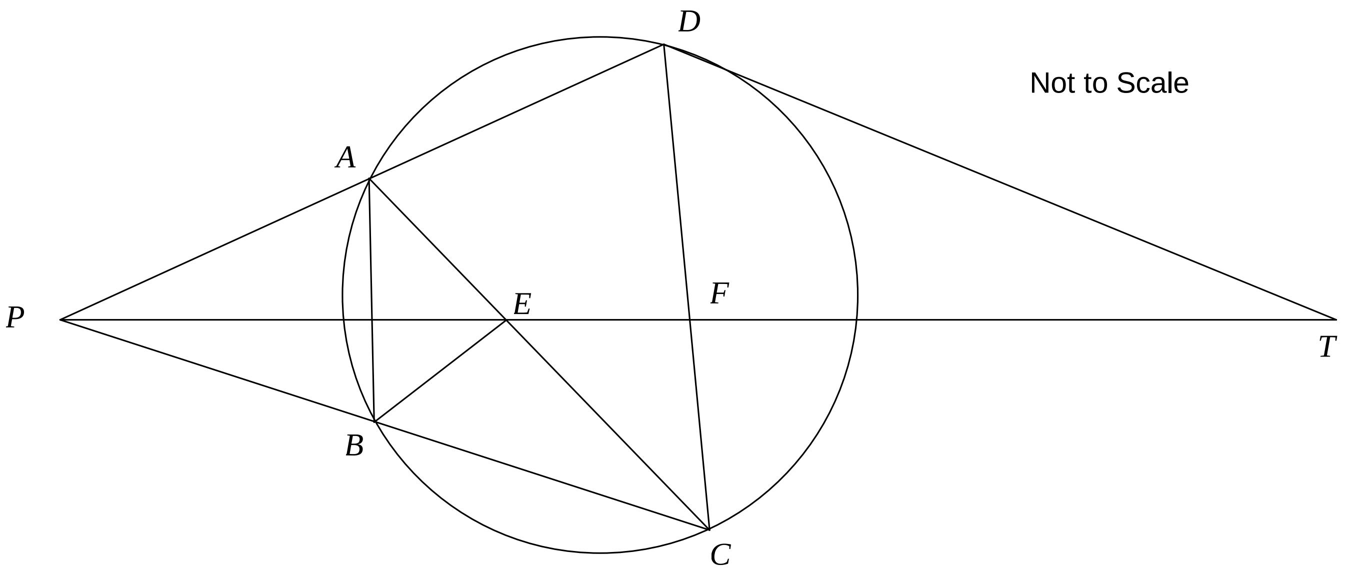
(iv) This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint of RS has coordinates $M\left(2t, -2t^3\right)$. **3**

(v) Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply. **2**

Marks

QUESTION 6 (15 marks)
Start a new writing booklet.

(a) $ABCD$ is a cyclic quadrilateral. DA produced and CB produced meet at P. T is a point on the tangent at D. PT cuts CA and CD at E and F respectively. $TF = TD$.



(i) Copy the diagram and show that $AEFD$ is a cyclic quadrilateral. **3**

(ii) Show that $AEBP$ is a cyclic quadrilateral. **2**

(b) (i) If $I_n = \int_0^1 \frac{dx}{\left(1+x^2\right)^n}$ prove that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$. **3**

(ii) Hence evaluate $\int_0^1 \frac{dx}{\left(1+x^2\right)^3}$. **2**

(c) (i) Use the substitution $x = a - y$ where a is a constant to prove that

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$ **1**

(ii) Hence show that $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$ **4**

Marks

QUESTION 7 (15 marks)
Start a new writing booklet.

(a) The functions $S(x)$ and $C(x)$ are defined by the formulae:

$$S(x) = \frac{1}{2}(e^{x} - e^{-x}) \text{ and } C(x) = \frac{1}{2}(e^{x} + e^{-x}).$$

(i) Verify that $S'(x) = C(x)$. **1**

(ii) Show that $S(x)$ is an increasing function for all real values of x. **1**

(iii) Prove that $\left[C(x)\right]^2 = 1 + \left[S(x)\right]^2$. **1**

(iv) $S(x)$ has an inverse function $S^{-1}(x)$ for all values of x. Briefly justify this statement. **1**

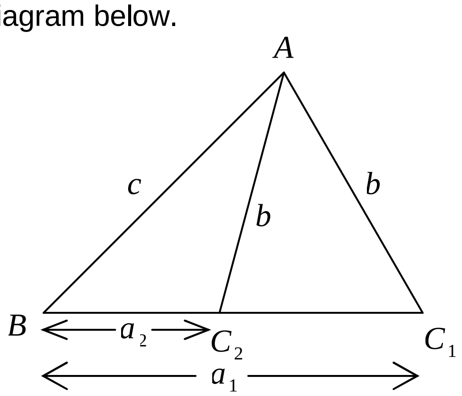
(v) Let $y = S^{-1}(x)$. Prove $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$. **2**

(vi) Hence, or otherwise, show that $S^{-1}(x) = \log_e\left(x + \sqrt{1+x^2}\right)$. **2**

(b) (i) Using the remainder theorem, or otherwise, show that $x - a - b - c$ is a factor of $P(x) = (x-a)(x-b)(x-c) - (b+c)(c+a)(a+b)$. **2**

(ii) Hence, or otherwise, solve the equation $(x-2)(x+3)(x+1) - 4 = 0$. **2**

(c) In $\triangle ABC$ the lengths b and c and $\angle B$ are given and have such values that two distinct triangles are possible as shown in the diagram below.



Show that $a_1 - a_2 = 2\sqrt{b^2 - c^2 \sin^2 B}$ **3**

QUESTION 8 (15 marks) **Marks**
Start a new writing booklet.

(a) A particle of mass $1\,kg$ moves in a straight line before coming to rest. The resultant force acting on the particle directly opposes its motion and has magnitude $m(1+v)$ where v is its velocity. Initially the particle is at the origin and travelling with velocity Q where $Q > 0$

(i) Show that v is related to the displacement x by the formula $x = Q - v + \log_e\left(\dfrac{1+v}{1+Q}\right)$. **3**

(ii) Find an expression for v in terms of t. **2**

(iii) Find an expression for x in terms of t. **1**

(iv) Show that $Q = x + v + t$ **1**

(v) Find the distance travelled and the time taken by the particle in coming to rest. **2**

(b) (i) State why, for $x < 1$, the sum of n terms of the series $1 + x + x^2 + x^3 + ... + x^{n-1}$ is $\dfrac{1-x^n}{1-x}$. **1**

(ii) Show that $1 + 2x + 3x^2 + ... + (n-1)x^{n-2} = \dfrac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$ **2**

(iii) Hence find an expression for $1 + 1 + \dfrac{3}{4} + \dfrac{4}{8} + ... + \dfrac{n-1}{2^{n-2}}$ and show that this sum is always less than 4. **3**

END OF PAPER

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \quad = \frac{1}{a}e^{ax}, \qquad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

$$NOTE: \quad \ln x = \log_e x, \qquad x > 0$$