Epping Boys High School

4 unit mathematics

Trial HSC Examination 1990

1. **(a)** Express in the form $x + iy$:
(i) $\frac{2+3i}{1-2i}$ **(ii)** Square root of $4 + 3i$
(b) Given that $\arg Z = \frac{\pi}{6}$, mod $Z = \sqrt{2}$
(i) Express Z in the form $x + iy$
(ii) Express Z^8 in $x + iy$ form.
(c) Find in mod-arg form all complex numbers z such that $z^3 = 1$. Plot them on an Argand Diagram.
(d) Solve $z\overline{z} + 2z = \frac{1}{4} + i$ for $z = x + iy$.
(e) On an Argand diagram shade in the region containing the points satisfying $|z| < 4$ or $\frac{\pi}{4} \leq \arg z \leq \frac{3\pi}{4}$.
(f) Draw a neat sketch of the locus $\Re(z) = |z - 2|$.

2. **(i)** Show that: **(a)** $\int_2^3 \frac{2x}{(x^4-1)}\,dx = \frac{1}{2}\ln\frac{4}{3}$
(b) $\int_0^2 \sqrt{\frac{x}{4-x}}\,dx = \pi - 2$ (Hint: let $x = 4\sin^2\theta$)
(ii) Find the following:
(a) $\int x^3 \log_3 x\,dx$
(b) $\int x^2\sqrt{1-x}\,dx$
(iii) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, show that $I_n + I_{n-2} = \frac{1}{n-1}$. Hence evaluate I_5.

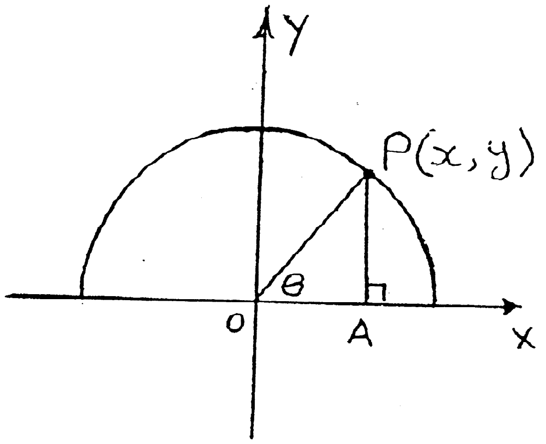
3. **(a)** Solve the equation $x^4 + x^3 - 8x^2 + 14x - 8 = 0$, being given that $1 + i$ is one root.
(b) Show that $x^3 + 4x - 1 = 0$ has only one real root, giving the integers between which this root lies.
(c) Show that if $P(x)$ has a zero α of multiplicity m, then $P'(x)$ has the zero α of multiplicity $(m - 1)$.
(d) Prove that if α is a zero of multiplicity m of the H.C.F. of $P(x)$ and its derivative $P'(x)$, then α is a zero of multiplicity $(m + 1)$ of $P(x)$.
(e) Determine whether $P(x) = 4x^3 + 4x^2 - 15x - 18$ has a repeated zero.
(f) Prove directly that if $\frac{a}{b}$ is in its lowest terms, and is a rational zero of $P(x) = 8x^4 - 5x^2 - 7x + 3$, then $b|8$ and $a|3$.
(g) Given that $\alpha_1, \alpha_2\alpha_3$, are the zeros of $2x^3 - 4x^2 - 3x - 1$, find:
(i) $\sum \alpha_i^2$ **(ii)** $\sum \alpha_i^3$ **(iii)** $\sum \alpha_i^4$
(iv) an equation whose roots are $\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_3$, $\alpha_2 + \alpha_3$.

4. **(i)** **(a)** If x is real show that $\frac{x^2+2x+1}{x+2}$ is either ≥ 2 or ≥ -4. (Don't have to use Calculus)
(b) Determine all the asymptotes of the curve $y = x + \frac{1}{x+2}$
(c) Draw a careful sketch of the curve $y = x + \frac{1}{x+2}$
(ii) Given the function $f(x) = x\sqrt{4 - x^2}$
(a) State its natural domain and show that it is an odd function.
(b) Show that on the curve $y = f(x)$ stationary points occur at $x = -\sqrt{2}, \sqrt{2}$ and determine their nature.
(c) Draw a neat sketch of the curve $y = f(x)$, indicating all the feature points.
(d) On different diagrams, sketch the curves **($\boldsymbol{\alpha}$)** $y = |f(x)|$ **($\boldsymbol{\beta}$)** $y^2 = x^2(4 - x^2)$

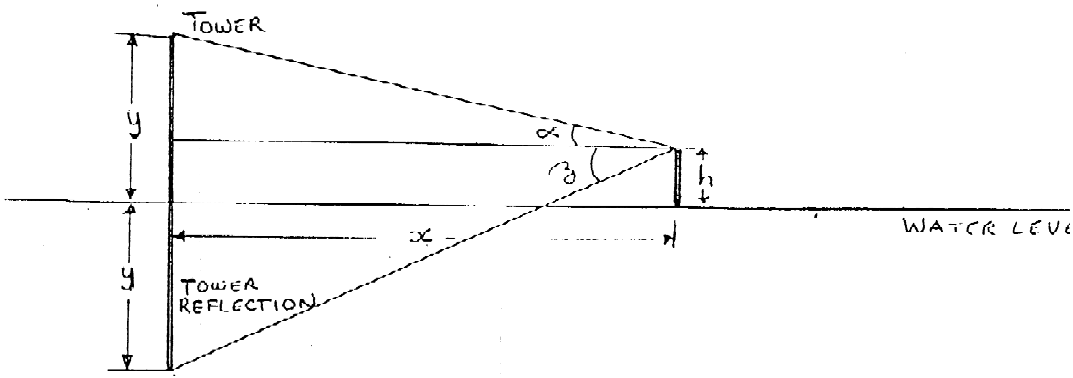
5. **(i)** Given the equations of circles $S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$,
$S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
(a) Find the centre and radius of S_1.
(b) Determine the conditions for $y = mx + c$ to be a common tangent to S_1 and S_2.
(c) Find the equation of the tangent common to both $x^2 + y^2 - 6x - 2y = 30,\ x^2 + y^2 - 9x - 3y = 0$.
(d) "The equation of a circle passing through the points of intersection of circles S_1 and S_2 is $S_1 + \lambda S_2 = 0$, where λ is a real constant". Justify this statement, and consider the case $\lambda = -1$.
(e) Given that $S_1 : x^2 + y^2 + 2x - 7 = 0$, $S_2 : x^2 + y^2 + 4x - 2y - 5 = 0$ intersect at the points P, Q:
($\boldsymbol{\alpha}$) Show that the equation of the common chord is $x - y + 1 = 0$;
($\boldsymbol{\beta}$) Find the length of PQ;
(γ) Determine the mid-point of PQ;
($\boldsymbol{\delta}$) Find the equation of the circle with PQ as its diameter.

6. **(i)** Find the general solution of the equation $\tan 2x = 2\sin x$

(ii) The point $P(x, y)$ lies on the semi-circle $y = \sqrt{4 - x^2}$. A is the foot of the ordinate of P and O is the origin. Show that $\angle POA = \theta = \tan^{-1} \frac{y}{\sqrt{4-y^2}}$. Find $\frac{d\theta}{dy}$ when $y = 1$.



(iii) A vertical tower stands on a river bank. From a point on the other bank directly opposite and at a height h above the water level, the angle of elevation of the top of the tower is α and the angle of depression of the reflection of the top of the tower is ζ. Prove that the width, x, of the river is $2h \cos\alpha \cos\zeta \operatorname{cosec}(\zeta - \alpha)$. The diagram below may be of some assistance to you. (Note: y is the height of the tower).



7. **(i)** P is a point $(a \sec\theta, b \tan\theta)$ on Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; whose eccentricity is e, an with foci S, S' (P lies in the first quadrant)

(a) Show that the equation of the tangent at P is $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$

(b) Show that $PS = a(e \sec\theta - 1)$ and hence $S'P - SP = 2a$

(c) The normal at P meets the transverse axis in G.
Show that $SG : SP = S'G : S'P = e$

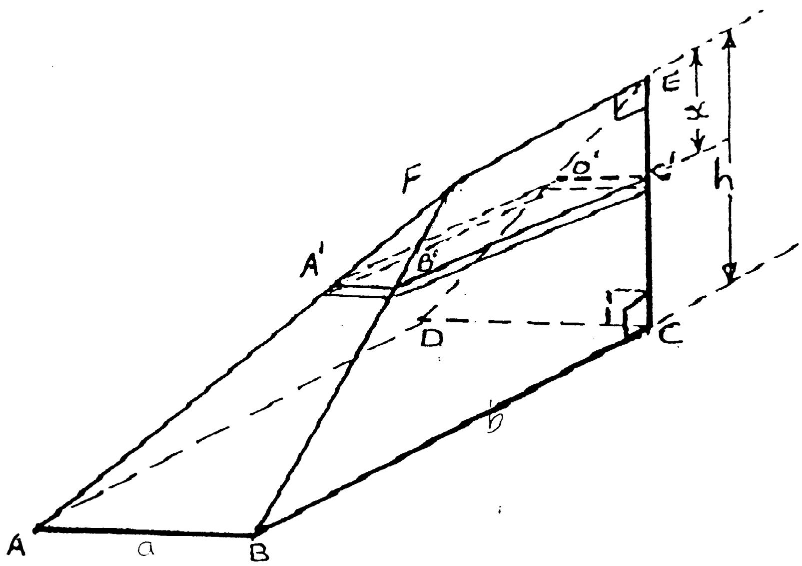
(d) This normal meets the conjugate axis in L. Show that the midpoint M of GL lies on the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = \frac{(a^2+b^2)^2}{4a^2b^2}$. Find the eccentricity of this hyperbola in terms of e.

(ii) In the complex plane show that the locus of z satisfying $|z + 3| + |z - 3| = 10$ is an ellipse.

(a) Find:

(α) The length of major and minor axis
(β) The equation of directrix
(γ) The coordinates of its foci
(b) Draw a neat sketch of the ellipse, clearly indicating all of the above.

8. (i) Consider solid $ABCDEF$ whose height is h, and whose base is a rectangle $ABCD$, where $AB = a$, $BC = b$, and the top edge $EF = c$.



Consider a rectangular slice $A'B'C'D'$ (parallel to the base $ABCD$) x units from the top edge, with width Δx.
NOTE: $B'C'||BC$ and $A'B'||AB$
(a) Show that the volume of the slice is $\Delta V = (\frac{x}{h}a)(c + \frac{b-c}{h}x)\Delta x$
(b) Hence show that the volume of the solid is $\frac{ha}{6}(2b + c)$
(ii) Shade in the region of the x-y-plane satisfied by each of the relations $0 \leq y \leq \sin x$, $0 \leq x \leq \pi$. Find the volume of the solid generated when this region is rotated through 2π about the line $x = \frac{\pi}{2}$.
(iii) A particle falls from rest at a height h above the ground. The retardation due to air resistance is KV^2, where K is a constant. Show that $V^2 = \frac{g}{k}(1 - e^{-2kx})$ hence find the greatest possible velocity.
(iv) A bird is 10m vertically above a man who throws a stone at an angle of projection of θ. The bird is flying with a uniform speed of 14m/sec. in a direction making $60°$ with the horizontal. Show that, for the stone to hit the bird, $\tan\theta \geq 2 + \sqrt{3}$.