Fort Street High School

4 unit mathematics

Trial HSC Examination 1986

1. (i) Sketch the following on the Argand diagram and describe in geometric terms the locus represented by:
(a) $|\frac{z-4}{z+3i}| = 1$ **(b)** $\arg(z+1-i) = \frac{\pi}{3}$
(ii) (a) State de Moivre's Theorem.
(b) Hence, prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$
(c) Solve the equation $\cos 5\theta = 1$ for $0 \le \theta < \pi$ and hence show that the roots of the equation $16x^5 - 20x^3 + 5x - 1 = 0$ are $x = \cos\frac{2k\pi}{5}$ for $k = 0, 1, 2, 3, 4$.
(d) Hence prove that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$ and $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$.
(iii) Solve the equation $z^6 + 1 = 0$, giving the roots in the form $a + ib$. Show these roots on an Argand diagram.
(iv) If $w = \frac{1+z}{1-z}$ and $|z| = 1$ where z and w are complex numbers, determine the locus of w.

2. (i) The ellipse E, is given in terms of the complex number z by: $|z+3| + |z-3| = 10$.
(a) Sketch E and determine the Cartesian equation of E.
(b) Prove that the area enclosed by E is 20π unit^2.
(ii) Prove that if z is a complex number then $\arg(\frac{z-i}{z+2}) = \frac{\pi}{2}$ represents the locus of a circle. Hence state the centre and radius of this circle.
(iii) Determine the factors of $6x^4 + 7x^3 + 21x^2 + 28x - 12$ over the field of
(a) rational numbers, $\mathbb{Q}$.
(b) complex numbers, $\mathbb{C}$.

3. (i) Decompose $\frac{6x^3-3x^2+22x-5}{(x-1)^2(x^2+9)}$ into partial fractions over the field of real numbers.
(ii) Write $\sqrt{5-12i}$ in the form $a + ib$, where a and b are real numbers.
(iii) (a) Find the coordinates of the foci and equations of the directrices and asymptotes of the hyperbola $5x^2 - 4y^2 = 20$. Sketch the curve.
(b) The tangent at a variable point P on this hyperbola meets a directrix at T. Show that PT subtends a right angle at the corresponding focus.
(iv) Prove that the polynomial $P(x) = \frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 4x + c$ has no real zeros if $c > 9\frac{1}{3}$.

4. (i) The curve $y = f(x)$ may be represented parametrically by: $x = \sin t - 1$ and $y = t - \cos t$.

(a) If the arc length of this curve between $t = 0$ and $t = \pi$ is given by: $L = \int_0^{\pi} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$ show that $L = \sqrt{2} \int_0^{\pi} \sqrt{1 + \sin t}\ dt$.
(b) Use seven evenly spaced ordinates from $t = 0$ to $t = \pi$ and Simpson's rule to estimate L to two decimal places.
(ii) Evaluate the following:
(a) $\int_{-\pi}^{\pi} \frac{\sin^5 x}{1+\cos^2 x}\ dx$ **(b)** $\int_0^{\pi} x \cos 2x\ dx$ **(c)** $\int_4^{\infty} \frac{dx}{16+4x^2}$

5. **(i)** Determine the following integrals:
(a) $\frac{(4 \tan x - 1) \sec^2 x\ dx}{(\tan x - 1)^2}$ **(b)** $\int \frac{dx}{3+4 \cos x}$ **(c)** $\int \frac{dx}{(3x^2-5x+4)^{\frac{1}{2}}}$ **(d)** $\int \operatorname{cosec}^3 x\ dx$.
(ii) If $I_n = \int x^n e^x\ dx$, prove that $I_n = x^n e^x - nI_{n-1}$. Hence evaluate $\int_0^1 x^3 e^x\ dx$.

6. **(a)** Outline Newton's Method for estimating a root r, of the equation $P(x) = 0$. In your answer include an appropriate diagram and derivation of the expression for the 2nd approximation z_2 of r in terms of the 1st approximation z_1.
(b) Use Newton's Method to estimate the first positive solution of $\tan x = -\frac{1}{x}$ correct to two decimal places.
(c) Sketch the curve $y = \frac{x}{\cos x}$ for $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$ using part **(b)** or otherwise. In your answer consider odd/even properties, vertical asymptotes, limits, stationary points, points of inflexion and the extreme values of the curve.

7. **(a)** The area bounded by the curve $y = 4x^2 - x^4$ and the x-axis between $x = 0$ and $x = 2$ is rotated about the y-axis. By slicing perpendicular to the y-axis show that the area of a cross-sectional slice is of the form $A(y) = 2\pi(4 - y)^{\frac{1}{2}}$. Hence calculate the volume of the solid generated.
(b) A solid sphere is formed by the rotation of the circle $x^2 + y^2 = 16$ about the y-axis (units are in cm). A cylindrical hole of diameter 4cm is bored through the centre of the sphere in the direction Oy.
(i) By considering a slice perpendicular to the x-axis use the method of cylindrical shells to determine the volume of the solid remaining.
(ii) Also determine the volume of the section cut out from the sphere.

8. **(a)** A sequence $u_1, u_2, u_3, \ldots$ is defined by the relations: $u_1 = 1, u_2 = 5$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for $n = 2, 3, \ldots$. Prove using the method of mathematical induction that $u_n = 3^n - 2^n$.
(b) In a triangle ABC the altitudes AD, BE and CF meet in the point H. The altitude AD also intersects the circumcircle of triangle ABC in X.
(i) Explain why $HDCE$ and $AEDB$ are cyclic quadrilaterals.
(ii) Prove that the triangles BDH and BDX are congruent.
(c) If $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ are acute show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$. Hence solve $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$.