



THE KING'S SCHOOL

2008
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Total marks – 120

Attempt Questions 1-8

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet.

Marks

(a) Find $\int \tan^2 x \, dx$ 2

(b) Find $\int \frac{x}{x+1} \, dx$ 2

(c) (i) Let $F(x)$ be a primitive function of $f(x)$. Hence, or otherwise, show that $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$ 1

(ii) Show that $\int_0^\pi x \sin x \, dx = \frac{\pi}{2} \int_0^\pi \sin x \, dx$ 2

(iii) Use integration by parts to evaluate $\int_0^\pi x^2 \cos x \, dx$ 3

(d) (i) Find $\int \frac{dt}{(2t+1)^2 + 1}$ 1

(ii) Use the substitution $t = \tan \frac{\theta}{2}$ to show that

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sin\theta - \cos\theta + 3} = \tan^{-1} \left(\frac{1}{2} \right)$$
 4

End of Question 1

(a) (i) Find $|\sqrt{7} + \sqrt{33} i|$ 1

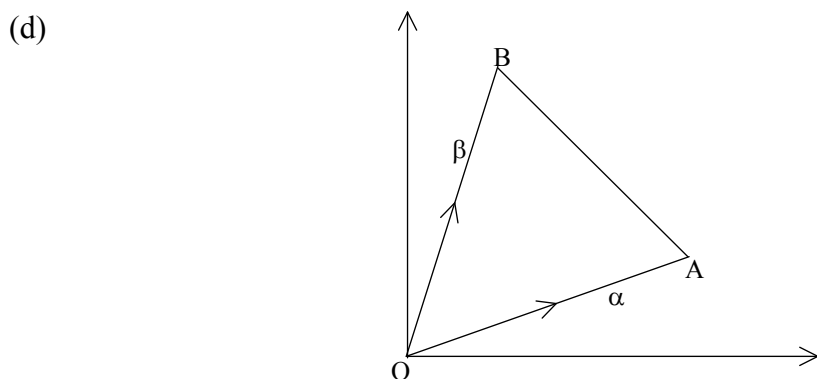
(ii) $x + iy = \frac{\sqrt{7} + \sqrt{33} i}{3 - i}$
 Find the value of $x^2 + y^2$ 2

(b) Precisely show on the Argand diagram the locus of the complex numbers z such that $|z - i| = 1$ and $|z| \leq 1$ hold simultaneously. 3

(c) Let $z = 1 - \cos 2\theta + i \sin 2\theta$, $0 < \theta < \frac{\pi}{2}$

(i) Show that $z = 2\sin\theta (\sin\theta + i\cos\theta)$ 2

(ii) Hence find $|z|$ and $\arg z$ 2



The diagram shows the equilateral triangle OAB in the complex plane.

O is the origin and points A, B represent the complex numbers α , β , respectively.

Let $v = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$

(i) Write down the complex number \overrightarrow{BA} 1

(ii) Show that $\alpha = v(\alpha - \beta)$ 2

(iii) Prove that $\alpha^2 + \beta^2 = \alpha\beta$ 2

End of Question 2

(a) For the hyperbola $(y + 1)^2 - x^2 = 1$, prove that $\frac{d^2y}{dx^2} = \frac{1}{(y + 1)^3}$ **4**

(b) Let $P(x) = x^4 - 2Ax^3 + B$, where $A \neq 0$

$P(x) = 0$ has the roots α, β, γ and $\alpha + \beta + \gamma$

(i) Deduce that $B = A^4$ **3**

(ii) Find, in simplest form, $\alpha^2 + \beta^2 + \gamma^2$ **3**

(c) Let $(1 + x)^{2008} = u_1 + u_2 + \dots + u_k + u_{k+1} + \dots + u_{2009}$, $x > 0$

(i) Show that $\frac{u_{k+1}}{u_k} = \frac{2009 - k}{k} \cdot x$ **2**

(ii) The middle term in the expansion of $(1 + x)^{2008}$ is the greatest term.

Deduce that $\frac{1004}{1005} < x < \frac{1005}{1004}$ **3**

End of Question 3

-
- (a) (i) Sketch the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ showing its foci, directrices and asymptotes. **4**

- (ii) A particular solid has as its base the region bounded by the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ and the line $x = 4$.

Cross-sections perpendicular to this base and the x axis are equilateral triangles.

Find the volume of this solid. **4**

- (b) A particle moves on the x axis according to the acceleration equation of motion $\ddot{x} = x$. Initially the particle is at the origin with velocity $v = 2$.

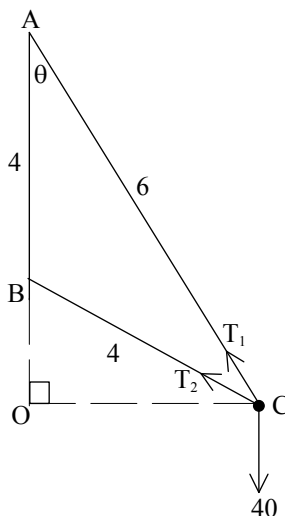
- (i) Explain why the velocity will always increase. **1**

- (ii) By integration, prove that $v = \sqrt{x^2 + 4}$ **2**

- (iii) By using the table of standard integrals, or otherwise, find the displacement x as a function of time t . **4**

End of Question 4

(a)



Two pieces of light inextensible string AC of length 6 metres and BC of length 4 metres are attached at two points A and B, respectively. B is 4 metres vertically below A.

At C a mass of 4 kg is attached to the strings and this mass rotates in uniform circular motion of 3 rad/s about a point O which is vertically below B. Take 10 m/s^2 as the acceleration due to gravity.

Let the tensions in the strings AC and BC be T_1 Newtons and T_2 Newtons, respectively, and let $\angle BAC = \theta$

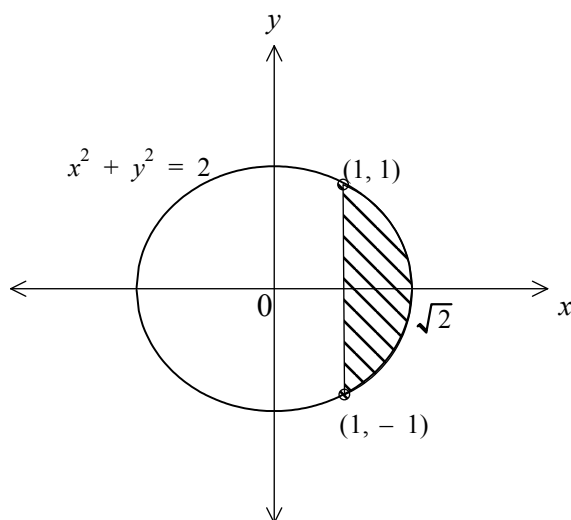
- (i) Show that $\cos \theta = \frac{3}{4}$ 1
- (ii) By resolving forces at C in the vertical direction, show that $6T_1 + T_2 = 320$ 3
- (iii) Find the tensions in the strings. 3

Question 5 continues on the next page

(b) (i) Show that $\int_1^{\sqrt{2}} x\sqrt{2-x^2} dx = \frac{1}{3}$ 2

(ii) By considering the circle $x^2 + y^2 = 2$, or otherwise, show that $\int_1^{\sqrt{2}} \sqrt{2-x^2} dx = \frac{\pi}{4} - \frac{1}{2}$ 2

(iii)

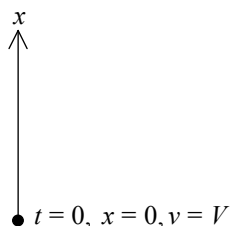


The minor segment of the circle $x^2 + y^2 = 2$ bounded by the chord $x = 1$ is revolved about that chord.

Use the method of cylindrical shells to find the volume of the solid generated. 4

End of Question 5

(a)



A particle of mass m is projected vertically upwards with speed V in a medium where there is a resistance mgk^2v^2 when v is its speed. g is the acceleration due to gravity and k is a positive constant.

Take $x = 0$ and $v = V$ when $t = 0$

The particle reaches a maximum height X when the time is T .

(i) Show that the equation of motion is given by $\ddot{x} = -g(1 + k^2v^2)$ 1

(ii) Show that $X = \frac{1}{2gk^2} \ln(1 + k^2V^2)$ 4

(iii) Show that $T = \frac{1}{gk} \tan^{-1}(kV)$ 3

(iv) If the only force acting on the particle is due to gravity the equations of motion are:

$$\ddot{x} = -g$$

$$\dot{x} = -gt + V$$

$$x = -\frac{gt^2}{2} + Vt$$

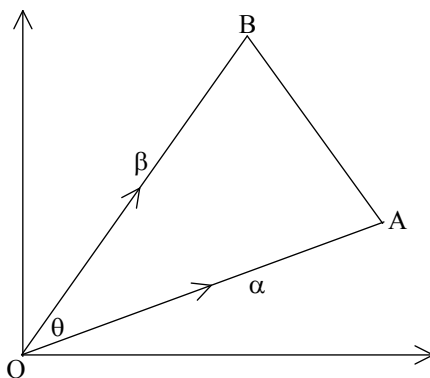
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Deduce that $\lim_{k \rightarrow 0} \frac{\ln(1 + k^2V^2)}{k^2} = V^2$ 2

Question 6 continues next page

- (b) (i) Use the results $z + \bar{z} = 2\text{Re}(z)$ and $|z|^2 = z\bar{z}$ for complex numbers z to show that $|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 = 2\text{Re}(\alpha\bar{\beta})$ 3

(ii)

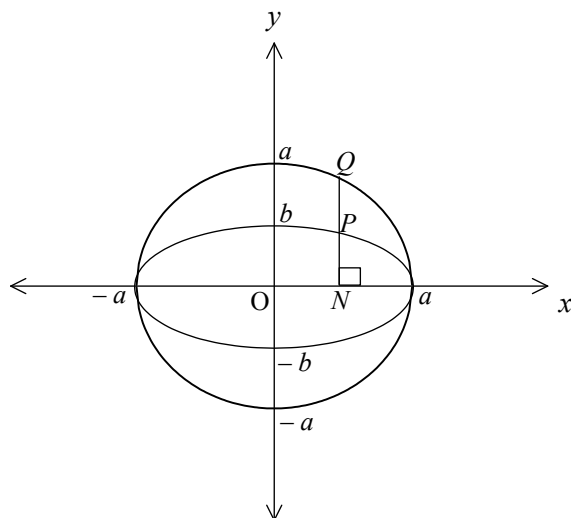


The diagram shows the angle θ between the complex numbers α and β .

Prove that $|\alpha| |\beta| \cos\theta = \text{Re}(\alpha\bar{\beta})$ 2

End of Question 6

(a)



The diagram shows the circle $x^2 + y^2 = a^2$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$

$P(a \cos\theta, b \sin\theta)$, $\theta \neq -\frac{\pi}{2}, \frac{\pi}{2}$, is a point on the ellipse. PN is perpendicular to the x axis at N and meets the circle at Q in the same quadrant.

O is the origin.

- (i) Write down the coordinates of Q . 1

- (ii) Show that the equation of the tangent at $P(a \cos\theta, b \sin\theta)$ on the ellipse is $\frac{\cos\theta}{a}x + \frac{\sin\theta}{b}y = 1$ 2

- (iii) Hence, or otherwise, find the equation of the tangent at the point Q on the circle. 1

- (iv) The tangents at P and Q meet at T . Prove that $ON \cdot OT = a^2$. 2

Question 7 continues on the next page

(b) Let $u_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$, $n = 0, 1, 2, \dots$

(i) Explain why $u_n < u_{n-1} < u_{n-2}$ 1

(ii) Prove that $u_n = \frac{n-1}{n} u_{n-2}$, $n = 2, 3, 4, \dots$ 3

(iii) Deduce that $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} u_{n-1}$ 1

(iv) Use (ii) to show that $n u_n u_{n-1} = \frac{\pi}{2}$, $n = 1, 2, 3, \dots$ 2

(v) Given that $\int_0^{\frac{\pi}{2}} \cos^{11} \theta \, d\theta = \frac{256}{693}$,
evaluate $\int_0^{\frac{\pi}{2}} \cos^{10} \theta \, d\theta$ 1

(vi) Find an approximate value of $\int_0^{\frac{\pi}{2}} \cos^{2008} \theta \, d\theta$ 1

End of Question 7

(a) Let $\frac{(x+a)(x+b)(x+c)}{(x-a)(x-b)(x-c)} \equiv k + \frac{p}{x-a} + \frac{q}{x-b} + \frac{t}{x-c}$

(i) Explain why $k = 1$ **1**

(ii) Show that $p = \frac{2a(a+b)(a+c)}{(a-b)(a-c)}$ and write down expressions for q and t . **3**

(iii) Hence, or otherwise, prove that

$$\frac{(a+b)(a+c)}{(a-b)(a-c)} + \frac{2b(b+c)}{(b-a)(b-c)} + \frac{2c(c+b)}{(c-a)(c-b)} = 1$$
 2

Question 8 continues on the next page

(b) The roots of $x^4 + x^3 + 2x^2 + 3x + 1 = 0$ are $\alpha, \beta, \gamma, \delta$

(i) Find a polynomial with the roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$. 2

(ii) Hence, or otherwise, show that

$$\left(\alpha + \frac{1}{\alpha}\right) + \left(\beta + \frac{1}{\beta}\right) + \left(\gamma + \frac{1}{\gamma}\right) + \left(\delta + \frac{1}{\delta}\right) = -4 \quad 1$$

(iii) Explain why the equation $x^4 + 4x^3 + Ax^2 + Bx + C = 0$ for some A, B, C has the roots $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}, \gamma + \frac{1}{\gamma}, \delta + \frac{1}{\delta}$ 1

(iv) Hence state the eight roots of the equation

$$\left(x + \frac{1}{x}\right)^4 + 4\left(x + \frac{1}{x}\right)^3 + A\left(x + \frac{1}{x}\right)^2 + B\left(x + \frac{1}{x}\right) + C = 0 \quad 1$$

(v) Use the equation in (iii) to state the four roots of the equation

$$Cx^4 + Bx^3 + Ax^2 + 4x + 1 = 0 \quad 1$$

(vi) By multiplying both sides by x^4 , the equation in (iv) could be expressed as $(x^2 + 1)^4 + 4x(x^2 + 1)^3 + Ax^2(x^2 + 1)^2 + Bx^3(x^2 + 1) + Cx^4 = 0$

[DO NOT SHOW THIS]

Hence, by using the polynomial found in (i) and another suitable equation, prove that $B = 0$. 2

(vii) Evaluate $\left(\alpha + \frac{1}{\alpha}\right)^{-1} + \left(\beta + \frac{1}{\beta}\right)^{-1} + \left(\gamma + \frac{1}{\gamma}\right)^{-1} + \left(\delta + \frac{1}{\delta}\right)^{-1}$ 1

End of Examination Paper

Standard Integrals

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

Note: $\ln x = \log_e x, \quad x > 0$



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Mathematics Extension 2

Question	(Marks)	Complex Numbers	Functions	Integration	Conics	Mechanics
1	(15)			15		
2	(15)	15				
3	(15)		15			
4	(15)			(a)(ii) 4	(a)(i) 4	(b) 7
5	(15)			(b) 8		(a) 7
6	(15)	(b) 5				(a) 10
7	(15)			(b) 9	(a) 6	
8	(15)		15			
Total	(120)	20	30	36	10	24

Question 1

$$(a) \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x$$

$$(b) \int \frac{x}{x+1} \, dx = \int \frac{x+1-1}{x+1} \, dx = \int 1 - \frac{1}{1+x} \, dx = x - \ln(1+x)$$

$$(c) \quad (i) \int_0^a f(a-x) \, dx = \left[-F(a-x) \right]_0^a = -F(0) + F(a) \\ = \int_0^a f(x) \, dx$$

$$(ii) \int_0^\pi x \sin x \, dx = \int_0^\pi (\pi-x) \sin(\pi-x) \, dx \\ = \int_0^\pi (\pi-x) \sin x \, dx \\ = \pi \int_0^\pi \sin x \, dx - \int_0^\pi x \sin x \, dx$$

$$\therefore 2 \int_0^\pi x \sin x \, dx = \pi \int_0^\pi \sin x \, dx \Rightarrow \int_0^\pi x \sin x \, dx = \frac{\pi}{2} \int_0^\pi \sin x \, dx$$

$$(iii) \int_0^\pi x^2 \cos x \, dx = \int_0^\pi x^2 \frac{d \sin x}{dx} \, dx \\ = \left[x^2 \sin x \right]_0^\pi - \int_0^\pi 2x \sin x \, dx \\ = 0 - \pi \int_0^\pi \sin x \, dx \quad \text{from (i)} \\ = \pi \left[\cos x \right]_0^\pi = \pi (-1 - 1) = -2\pi$$

$$(d) (i) \frac{1}{2} \tan^{-1}(2t+1)$$

$$(ii) t = \tan \frac{\theta}{2} \quad : \theta = 0, t = 0$$

$$\frac{dt}{d\theta} = \frac{1}{2} \sec^2 \frac{\theta}{2} = \frac{1}{2} (1+t^2) \quad \theta = \frac{\pi}{2}, t = 1$$

$$\therefore I = \int_0^1 \frac{2 dt}{(1+t^2) \left(\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} + 3 \right)}$$

$$= \int_0^1 \frac{2 dt}{4t - 1 + t^2 + 3 + 3t^2}$$

$$= \int_0^1 \frac{2 dt}{4t^2 + 4t + 2}$$

$$= \int_0^1 \frac{2 dt}{(2t+1)^2 + 1}$$

$$= \left[\tan^{-1}(2t+1) \right]_0^1$$

$$= \tan^{-1} 3 - \tan^{-1} 1$$

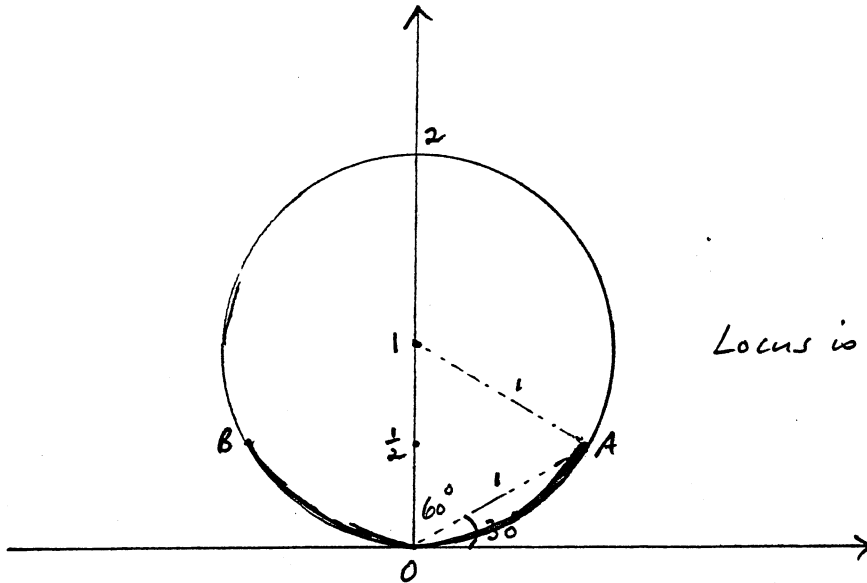
$$= \tan^{-1} \left(\frac{3-1}{1+3 \cdot 1} \right) = \tan^{-1} \frac{2}{4} = \tan^{-1} \frac{1}{2}$$

Question 2

$$(a) (i) = \sqrt{7+33} = \sqrt{40}$$

$$(ii) x^2 + y^2 = |x+iy|^2 = \frac{|\sqrt{7} + \sqrt{33}i|^2}{|3-i|^2} = \frac{40}{10} = 4$$

(b)



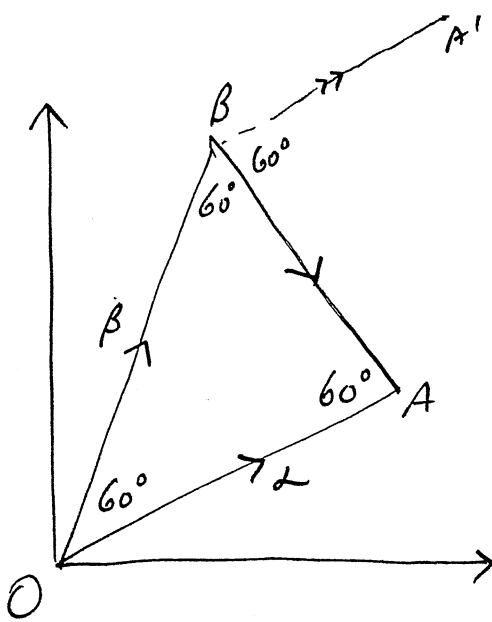
Locus is arc AB, $y \leq \frac{1}{2}$

$$(c) (i) z = 2 \sin \theta + i (2 \sin \theta \cos \theta) \\ = 2 \sin \theta (\sin \theta + i \cos \theta)$$

$$(ii) z = 2 \sin \theta \left(\cos \left(\frac{\pi}{2} - \theta \right) + i \sin \left(\frac{\pi}{2} - \theta \right) \right)$$

$$\Rightarrow |z| = 2 \sin \theta, \quad \arg z = \frac{\pi}{2} - \theta \quad \text{since } 0 < \theta < \frac{\pi}{2}$$

(d)



$$(i) \vec{BA} = \vec{l} - \vec{\beta}$$

$$(ii) v \vec{BA} = \vec{BA}' = \vec{l}, \text{ see diagram}$$

$$\text{i.e. } \vec{l} = v(\vec{l} - \vec{\beta})$$

$$(iii) \text{ Now, } \vec{\beta} = v \vec{l}$$

$$\therefore \frac{\vec{l}}{\vec{\beta}} = \frac{\vec{l} - \vec{\beta}}{\vec{l}}$$

$$\text{or } \vec{l} = \vec{l} - \vec{\beta}$$

$$\text{i.e. } \vec{l} + \vec{\beta} = \vec{l}$$

Question 3

$$(a) \quad 2(y+1)y' - 2x = 0$$

$$\therefore y' = \frac{x}{y+1}$$

$$\begin{aligned} \therefore y'' &= \frac{y+1 - x y'}{(y+1)^2} = \frac{y+1 - \frac{x^2}{y+1}}{(y+1)^2} \\ &= \frac{(y+1)^2 - x^2}{(y+1)^3} = \frac{1}{(y+1)^3} \end{aligned}$$

$$(b) \quad (i) \quad \Sigma \alpha = 2(\alpha + \beta + \gamma) = 2A \Rightarrow \alpha + \beta + \gamma = A$$

But, $\alpha + \beta + \gamma$ is a root

$$\therefore A^4 - 2A(A^3) + B = 0 \Rightarrow B = A^4$$

$$(ii) \quad \alpha^2 + \beta^2 + \gamma^2 + (\alpha + \beta + \gamma)^2 = (\Sigma \alpha)^2 - 2\Sigma \alpha\beta$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + A^2 = (2A)^2 - 2(0) = 4A^2$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = 3A^2$$

$$(c) \quad (i) \quad \frac{u_{k+1}}{u_k} = \frac{\binom{2008}{k} x^k}{\binom{2008}{k-1} x^{k-1}} = \frac{2008! (2009-k)! (k-1)!}{(2008-k)! k! 2008!} \cdot x$$

$$= \frac{2009-k}{k} \cdot x$$

(ii) The middle term is u_{1005}

$$\therefore u_{1005} > u_{1004}$$

$$\text{or } \frac{u_{1005}}{u_{1004}} > 1 \Rightarrow \frac{2009-1004}{1004} x > 1 \text{ from (i)}$$

$$\text{i.e. } x > \frac{1004}{1005}$$

$$\text{Also } u_{1005} > u_{1006}$$

$$\Rightarrow \frac{2009-1005}{1005} x < 1 \text{ from (i)}$$

$$\text{i.e. } x < \frac{1005}{1004}$$

$$\text{i.e. } \frac{1004}{1005} < x < \frac{1005}{1004}$$

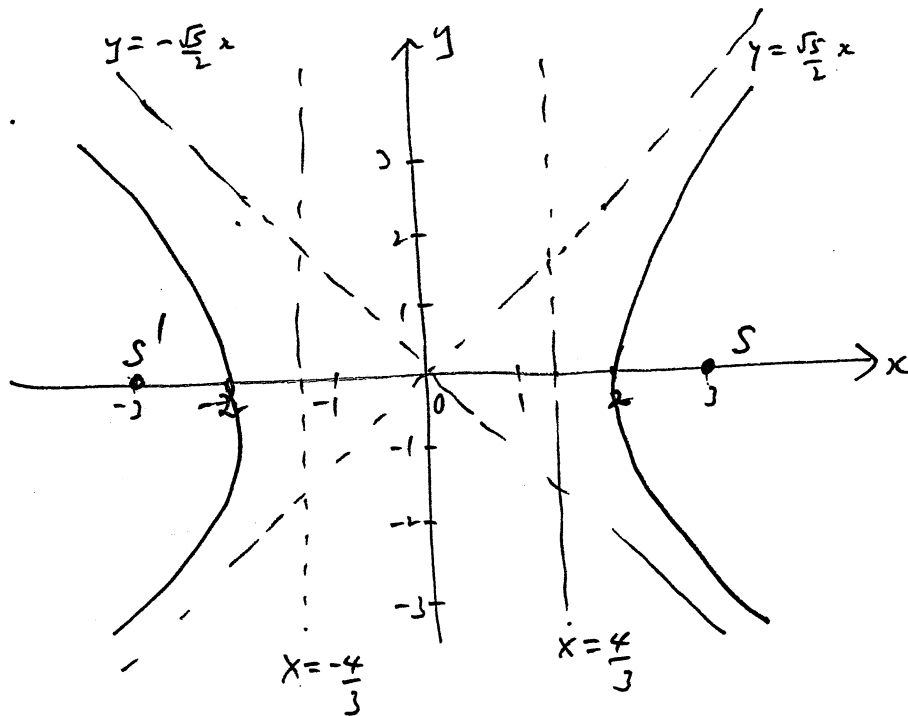
Question 4

$$(a) (i) c^2 = a^2 + b^2 \Rightarrow c^2 = 4 + 5 = 9, \quad c = 3$$

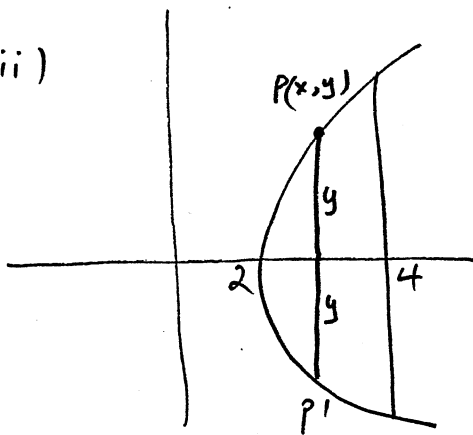
$$e = \frac{3}{2}$$

\therefore foci $(\pm 3, 0)$, directrices $x = \pm \frac{4}{3}$,

asymptotes $\frac{x}{2} \pm \frac{y}{\sqrt{5}} = 0$ i.e. $y = \pm \frac{\sqrt{5}}{2}x$



(ii)



Take $P(x, y)$ on curve, $y \geq 0$

Then, area Δ with base PP'

$$= \frac{1}{2} (2y)^2 \sin \frac{\pi}{3} = \sqrt{3} y^2$$

$$\therefore V = \int_2^4 \sqrt{3} y^2 dx$$

$$= \sqrt{3} \cdot 5 \int_2^4 \frac{x^2}{4} - 1 dx$$

$$= 5\sqrt{3} \left[\frac{x^3}{12} - x \right]_2^4$$

$$= 5\sqrt{3} \left(\frac{16}{3} - 4 - \frac{2}{3} + 2 \right)$$

$$= \frac{40\sqrt{3}}{3}$$

(b) (i) Initially $x=0$ and $v > 0$

\Rightarrow after $t=0$ then $x > 0$ i.e. $\ddot{x} > 0$

$\Rightarrow v$ will increase for all t

$$(ii) \frac{d(\frac{1}{2}v^2)}{dx} = x$$

$$\therefore \left[\frac{1}{2}v^2 \right]_0^x = \left[\frac{x^2}{2} \right]_0^x$$

$$\text{i.e. } \frac{1}{2}v^2 - \frac{2^2}{2} = \frac{x^2}{2}$$

$$\text{or } v^2 = x^2 + 4$$

$$\therefore v = \sqrt{x^2 + 4} \quad \text{since } v > 0$$

$$(iii) \frac{dx}{dt} = \sqrt{x^2 + 4}$$

$$\therefore \frac{dt}{dx} = \frac{1}{\sqrt{x^2 + 4}}$$

$$t = \int_0^x \frac{1}{\sqrt{x^2 + 4}} dx = \left[\ln(x + \sqrt{x^2 + 4}) \right]_0^x$$

$$\text{i.e. } t = \ln\left(\frac{x + \sqrt{x^2 + 4}}{2}\right)$$

$$\therefore \frac{x + \sqrt{x^2 + 4}}{2} = e^t$$

$$\text{or } \sqrt{x^2 + 4} = 2e^t - x$$

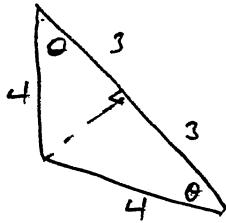
$$\therefore x^2 + 4 = 4e^{2t} - 4e^t x + x^2$$

$$\Rightarrow 1 = e^{2t} - e^t x$$

$$\text{or } x = \frac{e^{2t} - 1}{e^t} = e^t - e^{-t}$$

Question 5

(a) (i)



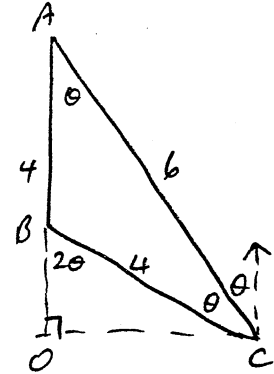
$$\Rightarrow \cos \theta = \frac{3}{4}$$

(ii) $T_1 \cos \theta + T_2 \cos 2\theta = 40$

$$\therefore T_1 \cos \theta + T_2 (2\cos^2 \theta - 1) = 40$$

$$\Rightarrow \frac{3}{4} T_1 + T_2 \left(\frac{9}{8} - 1 \right) = 40$$

$$\text{i.e. } 6T_1 + T_2 = 320$$



(iii) Resolving in the direction CO,

$$4 \cdot 3 \cdot OC = T_1 \cos\left(\frac{\pi}{2} - \theta\right) + T_2 \cos\left(\frac{\pi}{2} - 2\theta\right),$$

$$\text{where } OC = 4 \sin 2\theta$$

$$\therefore 16 \cdot 9 \cdot \sin 2\theta = T_1 \sin \theta + T_2 \sin 2\theta$$

$$\Rightarrow 16 \cdot 9 \cdot 2 \cos \theta = T_1 + T_2 \cdot 2 \cos \theta$$

$$\text{or } 216 = T_1 + T_2 \cdot \frac{3}{2}$$

$$\therefore \text{From (ii), } 6 \left(216 - \frac{3T_2}{2} \right) + T_2 = 320$$

$$1296 - 9T_2 + T_2 = 320$$

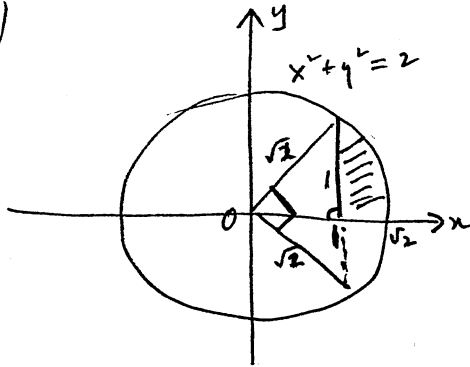
$$\text{or } T_2 = \frac{1296 - 320}{8} = 122 \text{ N}$$

$$\text{or } T_1 = 216 - \frac{3}{2} \times 122 \text{ N} = 33 \text{ N}$$

(b) (i) put $u = 2 - x^2$: $x = 1, u = 1$
 $\frac{du}{dx} = -2x$: $x = \sqrt{2}, u = 0$

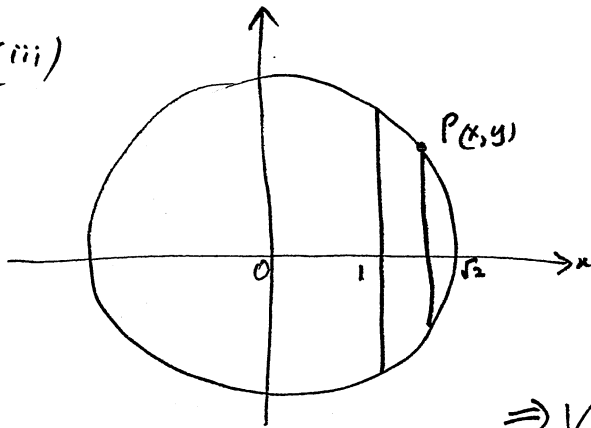
$$\therefore I = \frac{1}{2} \int_0^1 \sqrt{u} du = \frac{1}{3} [u^{3/2}]_0^1 = \frac{1}{3}$$

(ii)



\therefore Shaded area
 $= \int_1^{\sqrt{2}} \sqrt{2-x^2} dx$
 $= \frac{1}{2} \left[\frac{\pi(\sqrt{2})^2}{4} - \frac{1}{2}(\sqrt{2})^2 \right]$
 $= \frac{\pi}{4} - \frac{1}{2}$

(iii)



Take $P(x, y)$, $y \geq 0$, on circle

Then, $\delta V \approx \pi ((x+dx-1)^2 - (x-1)^2) 2y$
 $\approx 2\pi y \cdot 2(x-1)dx$

$$\Rightarrow V = 4\pi \int_1^{\sqrt{2}} (x-1)y dx$$

$$\therefore V = 4\pi \int_1^{\sqrt{2}} (x-1)\sqrt{2-x^2} dx$$

$$= 4\pi \int_1^{\sqrt{2}} x\sqrt{2-x^2} - \sqrt{2-x^2} dx$$

$$= 4\pi \left(\frac{1}{3} - \left(\frac{\pi}{4} - \frac{1}{2} \right) \right) \text{ from (i) + (ii)}$$

$$= 4\pi \left(\frac{5}{6} - \frac{\pi}{4} \right) = \frac{\pi}{3} (10 - 3\pi)$$

Question 6

$$(a) \quad (i) \quad m \ddot{x} = -mg - mgk^2 v^2$$

$$\Rightarrow \ddot{x} = -g(1 + k^2 v^2)$$

$$(ii) \quad \ddot{x} = v \frac{dv}{dx} = -g(1 + k^2 v^2)$$

$$\Rightarrow \frac{dv}{dx} = -g \left(\frac{1 + k^2 v^2}{v} \right)$$

$$\therefore -g \frac{dx}{dv} = \frac{v}{1 + k^2 v^2}$$

$$\therefore -g [x]_0^X = \int_V^0 \frac{v}{1 + k^2 v^2} dv$$

$$\begin{aligned} \text{i.e. } -gX &= \frac{1}{2k^2} \left[\ln(1 + k^2 v^2) \right]_V^0 \\ &= \frac{1}{2k^2} (0 - \ln(1 + k^2 V^2)) \end{aligned}$$

$$\therefore X = \frac{1}{2gk^2} \ln(1 + k^2 V^2)$$

$$(iii) \quad \ddot{x} = \frac{dv}{dt} = -g(1 + k^2 v^2)$$

$$\therefore -g \frac{dt}{dv} = \frac{1}{1 + k^2 v^2}$$

$$\Rightarrow -g [t]_0^T = \frac{1}{k} \left[\tan^{-1} kv \right]_V^0$$

$$\therefore -gT = \frac{1}{k} (0 - \tan^{-1} kV)$$

$$\therefore T = \frac{1}{gk} \tan^{-1}(kV)$$

$$(iv) \quad \dot{x} = 0 \Rightarrow x = \frac{V}{g}$$

$$\therefore x_{\max} = -\frac{g}{2} \cdot \frac{V^2}{g^2} + V \cdot \frac{V}{g} = \frac{V^2}{2g}$$

$$\therefore \text{from (ii), } \lim_{k \rightarrow 0} \frac{1}{2gk^2} \ln(1+k^2V^2) = \frac{V^2}{2g}$$

$$\text{i.e. } \lim_{k \rightarrow 0} \frac{\ln(1+k^2V^2)}{k^2} = V^2$$

$$(4) (i) \quad |\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2$$

$$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha - \beta)(\overline{\alpha - \beta})$$

$$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha - \beta)(\bar{\alpha} - \bar{\beta})$$

$$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha\bar{\alpha} - \alpha\bar{\beta} - \bar{\alpha}\beta + \beta\bar{\beta})$$

$$= \alpha\bar{\beta} + \bar{\alpha}\beta$$

$$= \alpha\bar{\beta} + \overline{(\alpha\bar{\beta})} = 2\operatorname{Re}(\alpha\bar{\beta})$$

$$(ii) \quad \cos \theta = \frac{|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2}{2|\alpha||\beta|} \quad \text{since } \vec{BA} = \alpha - \beta$$

$$= \frac{2\operatorname{Re}(\alpha\bar{\beta})}{2|\alpha||\beta|}$$

$$= \frac{2\operatorname{Re}(\alpha\bar{\beta})}{2|\alpha||\beta|}$$

$$\Rightarrow |\alpha||\beta|\cos \theta = \operatorname{Re}(\alpha\bar{\beta})$$

Question 7

(a) (i) $Q = (a \cos \theta, a \sin \theta)$

(ii) $\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$

$$= -\frac{b^2}{a^2} \cdot \frac{a \cos \theta}{b \sin \theta} \text{ at } P$$

$$= -\frac{b \cos \theta}{a \sin \theta}$$

\therefore tangent at P is $y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta} (x - a \cos \theta)$

$$\text{or } \frac{\sin \theta}{b} y - \sin^2 \theta = -\frac{\cos \theta}{a} x + \cos^2 \theta$$

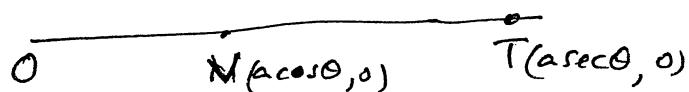
$$\text{i.e. } \frac{\cos \theta}{a} x + \frac{\sin \theta}{b} y = \cos^2 \theta + \sin^2 \theta = 1$$

(iii) (ii) \Rightarrow tangent at Q is $\frac{\cos \theta}{a} x + \frac{\sin \theta}{a} y = 1$ [if $a=b$]

(iv) at T , $\left(\frac{\sin \theta}{b} - \frac{\sin \theta}{a}\right) y = 0 \Rightarrow y = 0$

$$\therefore x = \frac{a}{\cos \theta} = a \sec \theta$$

i.e. we have



$$\therefore ON \cdot OT = |a \cos \theta| |a \sec \theta| = a^2$$

(b) (i) For $0 < x < \frac{\pi}{2}$, $0 < \cos x < 1$

$$\therefore \cos^n x < \cos^{n-1} x < \cos^{n-2} x$$

$$\Rightarrow u_n < u_{n-1} < u_{n-2}$$

$$(ii) u_n = \int_0^{\frac{\pi}{2}} \cos \theta \cos^{n-1} \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} \theta \frac{d \sin \theta}{d\theta} \, d\theta \quad \text{where } u = \cos^{n-1} \theta$$

$$= \left[\cos^{n-1} \theta \sin \theta \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \cos^{n-2} \theta \sin^2 \theta \, d\theta$$

$\frac{du}{d\theta} = -(n-1) \cos^{n-2} \theta \sin \theta$

$$= 0 + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} \theta (1 - \cos^2 \theta) \, d\theta$$

$$= (n-1) (u_{n-2} - u_n)$$

$$\therefore u_n (1 + n - 1) = (n-1) u_{n-2} \quad \text{i.e. } u_n = \frac{n-1}{n} u_{n-2}$$

$$(iii) u_n = \left(1 - \frac{1}{n}\right) u_{n-2}$$

$$\therefore \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} u_{n-2} \Rightarrow \text{from (i),}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} u_{n-1}$$

$$(iv) \text{ From (ii), } n u_n u_{n-1} = (n-1) u_{n-1} u_{n-2}$$

$$= (n-2) u_{n-2} u_{n-3}$$

$$= \dots$$

$$= 1 \cdot u_1 u_0$$

$$= \int_0^{\frac{\pi}{2}} \cos \theta \, d\theta \int_0^{\frac{\pi}{2}} 1 \, d\theta$$

$$= \left[\sin \theta \right]_0^{\frac{\pi}{2}} \left[\theta \right]_0^{\frac{\pi}{2}}$$

$$= 1 \cdot \frac{\pi}{2} = \frac{\pi}{2}$$

(v) From (iv), $11u_{11}u_{10} = \frac{\pi}{2}$

$$\therefore u_{10} = \frac{\pi}{22} \cdot \frac{693}{256} = \frac{63\pi}{512}$$

(vi) From (iii) and (iv), for large n ,

$$nu_n u_{n-1} \approx n u_n^2$$

$$\therefore 2008 u_{2008}^2 \approx \frac{\pi}{2}$$

$$\therefore u_{2008} \approx \sqrt{\frac{\pi}{4016}} \quad [\approx 0.028]$$

or, of course, $2009 u_{2008}^2 \approx \frac{\pi}{2}$

$$\Rightarrow u_{2008} \approx \sqrt{\frac{\pi}{4018}}$$

Indeed, $\sqrt{\frac{\pi}{4018}} < u_{2008} < \sqrt{\frac{\pi}{4016}}$

Question 8

(a) (i) $(x+a)(x+b)(x+c)$ and $(x-a)(x-b)(x-c)$ both have the same leading term $x^3 \Rightarrow k=1$

$$(ii) \frac{(x+a)(x+b)(x+c)}{(x-b)(x-c)} = x-a + p + \frac{q(x-a)}{x-b} + \frac{t(x-a)}{x-c}$$

$$\text{For } x=a, \frac{2a(a+b)(a+c)}{(a-b)(a-c)} = p$$

$$\therefore q = \frac{2b(b+a)(b+c)}{(b-a)(b-c)}$$

$$t = \frac{2c(c+a)(c+b)}{(c-a)(c-b)}, \text{ on symmetry.}$$

(iii) Put $x = -a$, then,

$$0 = 1 - \frac{p}{2a} - \frac{q}{a+b} - \frac{t}{a+c}$$

$$\Rightarrow 0 = 1 - \frac{(a+b)(a+c)}{(a-b)(a-c)} - \frac{2b(b+c)}{(b-a)(b-c)} - \frac{2c(c+b)}{(c-a)(c-b)}$$

$$\therefore \frac{(a+b)(a+c)}{(a-b)(a-c)} + \frac{2b(b+c)}{(b-a)(b-c)} + \frac{2c(c+b)}{(c-a)(c-b)} = 1$$

$$(b) (i) \left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right) + 1 = 0$$

$$\text{i.e. } x^4 + 3x^3 + 2x^2 + x + 1 = 0$$

$$(ii) \sum d = -1, \quad \sum \frac{1}{d} = -3$$

$$\therefore \sum \left(d + \frac{1}{d}\right) = -4$$

(iii) The sum of the roots of the quartic is -4

\therefore result from (ii)

$$(iv) d, \frac{1}{d}, \beta, \frac{1}{\beta}, \gamma, \frac{1}{\gamma}, \delta, \frac{1}{\delta}$$

$$(v) \left(d + \frac{1}{d}\right)^{-1}, \left(\beta + \frac{1}{\beta}\right)^{-1}, \left(\gamma + \frac{1}{\gamma}\right)^{-1}, \left(\delta + \frac{1}{\delta}\right)^{-1}$$

(vi) $(x^4 + x^3 + 2x^2 + 3x + 1)(x^4 + 3x^3 + 2x^2 + x + 1) = 0$ has
the roots $d, \frac{1}{d}, \dots, \delta, \frac{1}{\delta}$

$$\equiv (x^2 + 1)^4 + 4x(x^2 + 1)^3 + Ax^2(x^2 + 1)^2 + Bx^3(x^2 + 1) + Cx^4,$$

from (iv)

\therefore Equating coefficients of x^5 ,

$$1 + 2 + 6 + 3 = 12 + B$$

$$\therefore B = 0$$

(vii) From (v), $\sum \left(d + \frac{1}{d}\right)^{-1} = -\frac{B}{C} = 0$ from (vi)

* Note $C \neq 0$ since $d + \frac{1}{d} \neq 0$