Newington College

4 unit mathematics

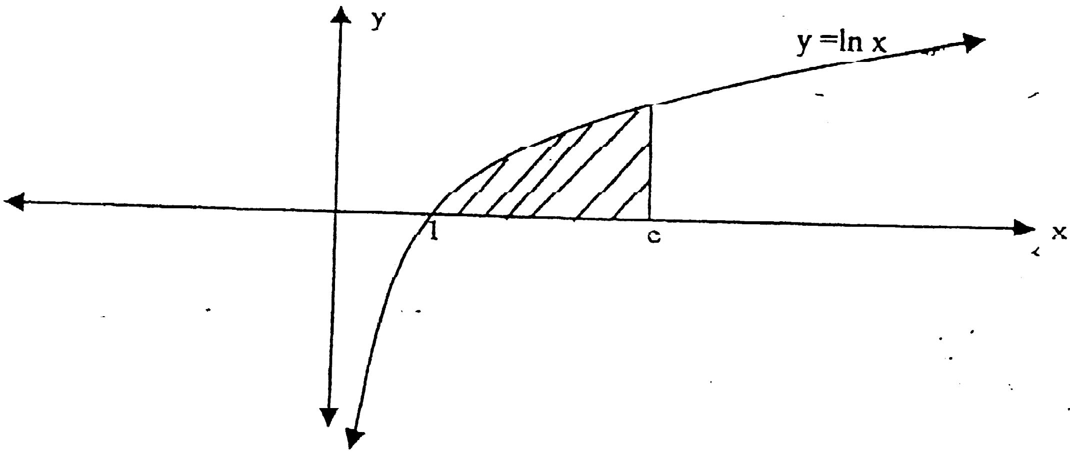
Trial HSC Examination 1999

1. **(a)** Evaluate **(i)** $\int_0^4 \frac{1}{\sqrt{x^2+9}}\, dx$ **(ii)** $\int_2^4 \frac{dx}{x^2-2x+4}$
(b) Find **(i)** $\int \frac{(\sqrt{x}-1)^6}{\sqrt{x}}\, dx$ **(ii)** $\int e^{-x} \cos\frac{x}{2}\, dx$
(c) **(i)** If $I_n = \int_0^{\frac{\pi}{4}} \tan^{2n} x\, dx$ prove that $I_n + I_{n-1} = \frac{1}{2n-1}$, for $n \geq 1$.
(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x\, dx$.

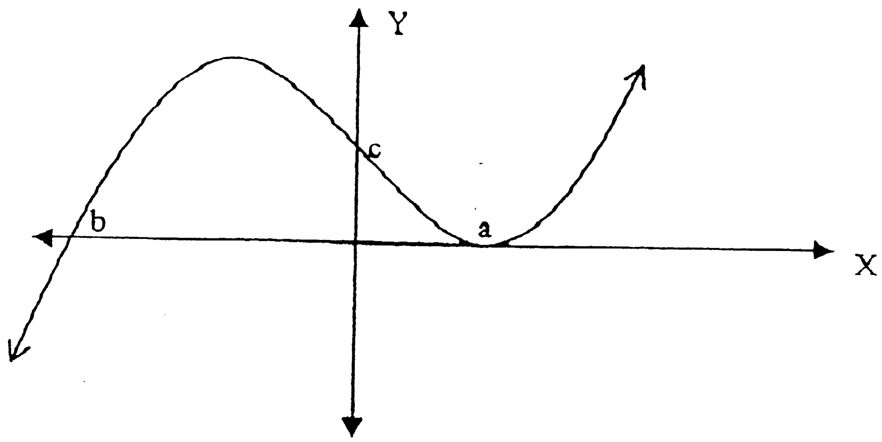
2. **(a)** **(i)** If $Z = -1 + \sqrt{3}i$, find $|Z|$ and $\arg Z$.
(ii) Hence evaluate $(-1 + \sqrt{3}i)^9$
(b) **(i)** Express the value of $(-1 + \sqrt{3}i)(1 + i)$ in the form $a + ib$.
(ii) Hence, or otherwise, find the exact value of $\cos\frac{11\pi}{12}$.
(c) Graph the region in the complex plane for which $2 < |z - 1 + 2i| < 3$.
(d) If $|z| < \frac{1}{2}$, show that $|(1 + i)z^3 + iz| < \frac{3}{4}$.

3. **(a)** Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$
(i) If $P(x)$ has the zeros $a + ib$, $a - 2bi$, where a and b are real, find the values of a and b.
(ii) Hence, find all the zeros of $P(x)$ over the complex field and express $P(x)$ as the product of two factors.
(b) **(i)** If α is a double root of $f(x) = 0$, show that α is a root of $f'(x) = 0$.
(ii) Show that if the equation $x^n + px + q = 0$ has a double root α (where α, p, q are real non-zero constants, and n is an integer with $n \geq 2$), then $\alpha = \frac{qn}{p(1-n)}$.

(c) Using the method of cylindrical shells, find the volume generated by revolving the region bounded by $y = \ln x$, the x-axis and $1 \le x \le e$, about the y-axis.



4. (a)



The graph of the function $y = f(x)$ is sketched above. On separate number planes sketch the graphs of:

(i) $y = f(-x)$ **(ii)** $y^2 = f(x)$ **(iii)** $y = f(|x|)$ **(iv)** $y = \frac{1}{1-f(x)}$

(b) (i) Resolve $\frac{1}{(x+1)(x^2+4)}$ into partial fractions.

(ii) Use this result to show that $\int_0^2 \frac{1}{(x+1)(x^2+4)}\, dx = \frac{1}{10}(\frac{\pi}{4} + \ln\frac{9}{2})$.

5. (a) (i) Consider the rectangular hyperbola $xy = c^2$, where $c > 0$. Prove that the equation of the chord joining the points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$, where $0 < p < q$, is given by $x + pqy = c(p+q)$.

(ii) The chord PQ intersects the x and y axes in A and B respectively. Prove that $AP = BQ$.

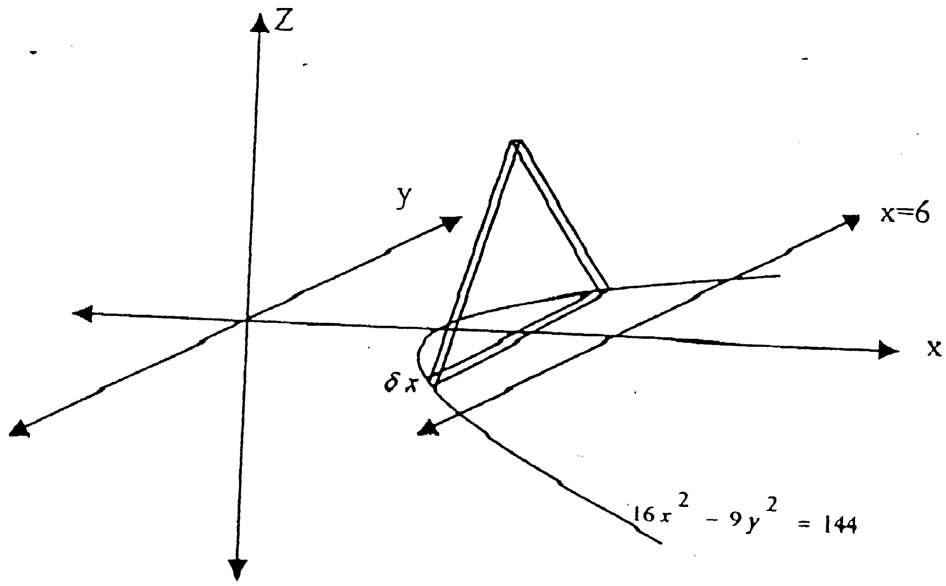
(iii) Show that the area enclosed by the hyperbola $xy = c^2$ and the chord PQ is

$\frac{c^2(q^2-p^2)}{2pq} + c^2 \ln \frac{p}{q}$ square units.

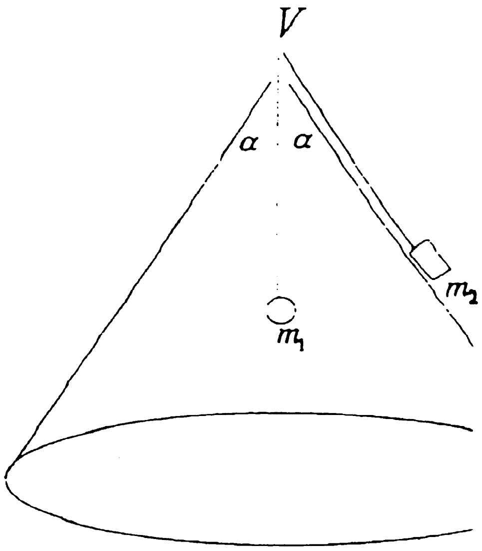
(b) A solid has as its base the area bounded by the hyperbola $16x^2 - 9y^2 = 144$ and the line $x = 6$. Every cross-section of this solid perpendicular to the x-axis is an isosceles triangle of altitude 3.

(i) Show that the volume V of the resulting solid is given by $V = 4\int_1^6 \sqrt{x^2 - 9}\, dx$.

(ii) Hence, show that $V = 36\sqrt{3} - 18\ln(2 + \sqrt{3})$



6. (a)



(7 marks)

A hollow cone whose vertical angle is 2α is fixed with its vertical and with vertex V uppermost. A light inextensible string passes without friction through a small hole at V and carries a particle P_1 of mass m_1 kg at one end so that P_1 hangs vertically at rest inside the cone. The other end of the string carries a particle P_2 of mass m_2 kg, which moves in a horizontal circle at constant angular velocity ω on the smooth outer surface of the cone, at a vertical depth h metres below V.

(i) Prove $m_2(h\omega^2 \sin^2 \alpha + g\cos^2 \alpha) = m_1 g \cos \alpha$.

(ii) Find the magnitude of the force exerted by the surface of the cone on P_2, and hence deduce that $h\omega^2 < g$.

(b) Two particles move in the same vertical line in a medium whose resistance, per unit mass, varies as the velocity. One particle is projected vertically upwards from the ground with initial velocity u, and starting at the same instant, the other particle falls from a height, h metres.

(i) For the particle which is projected vertically upwards from the ground, show that the expression for its height x metres after a time t seconds is given by $x = \frac{g+ku}{k^2}(1 - e^{-kt}) - \frac{gt}{k}$, where g is the acceleration due to gravity and k is a constant.

(ii) Assuming that the height of the falling particle is given by $h - \frac{gt}{k} - \frac{ge^{-kt}}{k^2} + \frac{g}{k^2}$, prove that the particles meet after a time, T, where $T = \frac{1}{k}\ln(\frac{u}{u-kh})$.

7. (a) A group of men and women is seated randomly around a circular table. What is the probability that none of the men are sitting next to each other if there are:

(i) 3 men and 4 women?

(ii) n men and $n+1$ women?

(b) If a, b, c and d are positive real numbers, prove that

(i) $\frac{a+b}{2} \geq \sqrt{ab}$,
(ii) $(a+b+c+d)^2 \geq 4(ac+bc+bd+ad)$,
(iii) $(a+b+c+d)^2 \geq \frac{8}{3}(ab+ad+bc+cd+bd+ac)$
(c) A sequence is defined by the relationship $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$ where $a_1 = 1$ and n is a positive integer.
(i) Show, using mathematical induction, that $\frac{a_n-\sqrt{2}}{a_n+\sqrt{2}} = (\frac{1-\sqrt{2}}{1+\sqrt{2}})^{2^{n-1}}$.
(ii) Hence find the limiting value of a_n as n becomes large.

8. (a) Find all real x such that $3\sqrt{x(1-x)} < |x-2|$
(b) If a curve is given by $y = f(x)$, where $f(x)$ has a continuous derivative in the open interval between $x = a$ and $x = b$ then the length is given by $\int_a^b [1+(\frac{dy}{dx})^2]^{\frac{1}{2}}\, dx$. Use this result to prove that the circumference of a circle, with radius r, is equal to $2\pi r$.
(c) (i) Prove that for $t \neq -1$, $1 - t + t^2 - t^3 + \cdots + t^{2n} = \frac{1}{1+t} + \frac{t^{2n+1}}{1+t}$
(ii) Hence deduce that for $x > -1$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{t^{2n+1}}{1+t}\, dt.$$

(iii) For $0 \leq x \leq 1$, find $\lim_{n\to\infty} \int_0^x \frac{t^{2n+1}}{1+t}\, dt$, giving reasons for your answer.
(iv) Hence find an infinite series converging to $\ln 2$.