PETRUS KY COLLEGE
NEW SOUTH WALES

in partnership with

VIETNAMESE COMMUNITY IN AUSTRALIA
NSW CHAPTER

JULY 2006

MATHEMATICS EXTENSION 2

PRE-TRIAL TEST

HIGHER SCHOOL CERTIFICATE (HSC)

Student Number: | | | | |

Student Name: ____________________

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Write your Student Number and your Name on all working booklets

Total marks – 96

- Attempt Questions 1–8
- Question 8 is optional
- All questions are of equal value

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Marks

Question 1 **12**

(A) Integrate the following,

(i) $\int \frac{\sin 2x}{\sqrt{1-\cos 2x}}\,dx$ 1

(ii) $\int \frac{dx}{x\sqrt{x^6-4}}$ (Let $x^3 = 2\sec u$) 2

(iii) $\int_{-1}^{1} \frac{2x}{(x^2+2x+5)^2}\,dx$ 2

(iv) $\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$ 2

(B) By substituting x = a – y , show that 1

$$\int_o^a f(x)dx = \int_0^a f(a-x)dx$$

Hence use this result to evaluate.

(i) $\int_o^1 x(1-x)^{12}\,dx$ 2

(ii) $\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\,dx$ 2

Question 2 **12**

(A) Given $Z_1 = i\sqrt{2}$ and $Z_2 = \dfrac{2}{1-i}$ 4

(i) Express Z_1 and Z_2 in the modulus/argument form.

(ii) If $Z_1 = w.Z_2$ express w in the modulus/argument form.

(iii) Show Z_1, Z_2 and $Z_1 + Z_2$ on an Argand diagram.

Hence show that

$$Arg(Z_1 + Z_2) = \frac{3\pi}{8}$$

Use the diagram to find the exact value of $\tan\dfrac{3\pi}{8}$

(B) If Z_1, Z_2 are complex numbers, prove that 4

$$\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$$

Given a complex number $Z = \dfrac{c+2i}{c-2i}$ where c is real.

Find $|Z|$ and hence describe the exact locus of Z if c varies from -1 to 1.

(C) If $w = 2\sqrt{3}i - 2$, find |w| and arg w, then indicate on an Argand diagram the complex number w, $\overline{w}$, iw, $\dfrac{1}{w}$, -w. 4

Show that $w^2 = 4\overline{w}$

Prove that w is a root of the equation $Z^3 - 64 = 0$. Find other roots.

Question 3 **12**

(A) P is the point ($3\cos\theta$, $2\sin\theta$) and Q is the point($3\sec\theta$, $2\tan\theta$).

Sketch the curves which are the loci, as θ varies, of P and Q, marking on them the range of positions occupied by P and Q respectively as θ varies from $\frac{\pi}{2}$ to π.

(i) Prove that for any value of θ, the line PQ passes through one of the common points of the 2 curves. 2

(ii) Show that the tangent at P to the first curve meets the tangent at Q to the second curve in a point which lies on the common tangent to the two curves at their other common point. 2

(iii) Prove that the two curves have the same length of the latus rectum. 2

(B) Show that the condition for a straight line y = mx + c to touch the ellipse E of 3

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=b^2+a^2m^2$

Hence show that the locus of the point P(x, y) from which the 2 tangents to the ellipse E

$\frac{x^2}{16}+\frac{y^2}{9}=1$ are perpendicular together, is a curve with the centre at the origin and a radius of 5.

Question 4 **12**

(A) A function is defined with the polar equation as follows:

$$\begin{cases} x = 8\cos^3\theta \\ y = 8\sin^3\theta \end{cases} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

(i) Find $\frac{dy}{dx}$ in term of θ and show that the graph of this function touches the x and y axis. Sketch the curve. 3

(ii) Show that the Cartesian equation of that function is 3

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

Show that the equation of the tangent to the curve at the point P(x_o, y_o) is:

$$y_0^{1/3}x + x_0^{1/3}y = 4x_0^{1/3}.y_0^{1/3}$$

Prove that the segment intercepted on this tangent by the coordinate axis is independent of the position of P on the curve.

(B) Consider the function $f(x) = 2 - \frac{4x}{x^2+1}$

(i) Show that the function is always positive for any value of x. 1

(ii) Find the asymptote (if any) and the stationary point of that curve. 1

(iii) Sketch the curve y = f(x). 1

(iv) On a separate diagram, sketch the relating curves:

a) y = f(|x|) 1

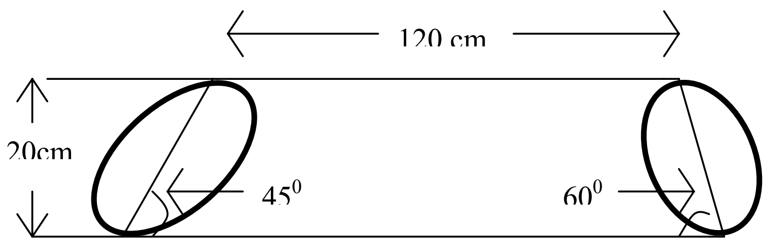
b) $y = \frac{1}{f(x)}$ 1

c) y = ln f(x) 1

Question 5 **12**

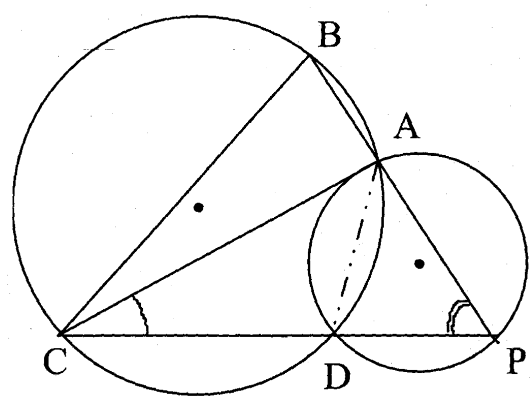
(A) The area bounded by the curve y=sinx, the two lines y=-x and x=π is rotated about the line y=-x. Find the volume of the solid shape of that formation. 6

(B) A cylindrical timber is chopped at two ends by 2 planes which are inclined 60^0 and 45^0 respectively. If the two ends are tilt toward each other and the shortest length of 2 ends is 120cm. Find the volume of that timber. Give the radius of timer is 10cm. 6



Question 6 **12**

(A)



Two circles intersect at A and D. P is the point on the major arc of one circle. The other circle has the radius r. PA produced and PD produced meet the other circle at B and C respectively. Let $\angle$APD = α and $\angle$ACD = β.

(i) Show that BC = 2 r sin ($\alpha + \beta$). 2

(ii) As P moves along the major arc AD on its circle, show that the length of the chord BC is independent of the position of P. 2

(iii) If the 2 circles have equal radii, show that 2

BC = 2 cosα .AD

(B) P is any point (ct, c/t) on the Hyperbola $xy = c^2$, whose centre is O.

(i) M and N are perpendicular roots of P to the 2 asymptotes. Prove that PM.PN is constant. 1

(ii) Find the equation of tangent at P, and show that OP and this tangent are equally inclined to the asymptotes. 2

(iii) If the tangent at P meet the asymptotes at A and B, and Q is the fourth vertex of the rectangle OAQB, find the locus of Q. 1

(iv) Show that PA=PB and hence conclude that the area of ΔOAB is independent of position of P. 2

Question 7 **12**

(A) If the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has two zeros $(a + ib)$ and $(a - 2ib)$ where a and b are real, then find the values of a and b. **4**

Hence find the zeros of P(x) over the complex field C, and express P(x) as the product of 2 quadratic factors with rational coefficients.

(B) Show that if the polynomial $P(x) = 0$ has a root a of multiplicity m, then $P'(x)\ 0$ has a root α of multiplicity $(m - 1)$. **4**

Given that $P(x) = x^4 + x^3 - 3x^2 - 5x —2= 0$ has a 3-fold root, find all the roots of P(x).

(C) Find the cubic roots of unity and express them in the form $r(\cos\theta + i\sin\theta)$.

Show these roots on an Argand diagram.

If w is one of the complex roots, prove that the other root is w^2 and show that $1 + w + w^2 = 0$.

(i) Prove that if n is a positive integer, then $1 + w^n + w^{2n}$ = **3 or 0** depending on whether n **is or is not** a multiply of 3. **2**

(ii) If $x = a + b$, $y = aw+ bw^2$ and $z = aw^2 + bw$, show that **2**

$z^2 + y^2 + x^2 = 6ab$

Question 8 - Optional **12**

(A) A particle is projected vertically upward with initial speed u. The air resistance is proportional to the speed of the particle.

(a) If $\ddot{x} = -(g + kw)$ with k is the constant, then find the maximum height reached by the particle and the time to do so. 2

.

(b) Set up the differential equation for the downward motion. 2

(c) Show that the particle returns to its point of projection with speed v given by 2

$$k(u+v) = g\log_e[\frac{g+ku}{-g-kV}]$$

(B) Show that for $n \ge 1$ 3

$$1.\ln\frac{2}{1} + 2\ln\frac{3}{2} + 3\ln\frac{4}{3} + ... + n\ln(\frac{n+1}{n}) = \ln(\frac{(n+1)^n}{n!}$$

(C) By using the induction method, prove that 3

$(35)^n + 3\times 7^n + 3\times 5^n + 6$ is divisible by 12 for $n \ge 1$

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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