9 Balfour Street
Greenwich NSW 2065
billpender@optusnet.com.au
Wednesday 31st August 2016

Tom Alegounarias
President, New South Wales Board of Studies, Teaching and Educational Standards NSW
GPO Box 5300
Sydney NSW 2001
brigit.workman@bostes.nsw.edu.au

Dear Mr Alegounarias,

Submission on the Board's 2016 Draft Calculus Courses

It is sad to see the Board issue yet another calculus-writing attempt that displays so little knowledge of mathematics, so little understanding of the secondary classroom, and so little respect for the excellent traditions of mathematics teaching in NSW and its foundations in arithmetic and geometry. The current Drafts are the worst in a long sequence of proposals over about twenty-five years, and by issuing them, the Board is in danger of losing its credibility as the authority responsible for mathematics education in NSW.

I have written to you many times over the years to give detailed criticisms of drafts, and I will do so again on this occasion, although this time I can only be selective because of the sheer multitude of errors. I will write at length, however, about the most fundamental matters — the language and structure of mathematics, the nature of mathematical understanding, the role of geometry, the way in which calculus is developed, and how a senior mathematics classroom works — because the Drafts have failed most comprehensively in these most basic tasks.

The criticisms will be familiar because you have received them from me and from many others over the years. There is nothing in these documents that makes a worthwhile new contribution to the teaching of senior mathematics in NSW, and there are problems in almost every aspect.

My recommendation is that you abandon these Drafts completely as any basis of future courses, that you abandon the attempt to align our courses either with ACARA guidelines or with Victorian material, that you take the present 2/3/4 Unit calculus courses as the basis for any future review, and that you use mathematicians nominated by the mathematics departments of our leading NSW universities to carry through any changes that you want to make. It is the job of teachers to make calculus appear straightforward in the classroom, but actually it is extremely tricky in all its details, and the formulation of a senior school syllabus needs profound knowledge of mathematics. I refer you here to my previous praise of the extraordinary work of Sydney University's Professor Room, who originally put our present excellent courses together.

The short time given to the consultation process, and its placement at the end of Term III when subject masters and directors of studies have their highest workload, is leading to questions as to how serious the Board is about its consultation process.

I will now set about justifying my remarks by a detailed critique of the Drafts. My remarks are most detailed on the 2 Unit course, because it should be the foundation on which the other two courses are built. To refer quickly to these three courses, whose names are still fluid, I will call them 'the calculus courses' or just 'the courses', and use the old terms '2 Unit', '3 Unit' and '4 Unit' as well as'Extension 1' and 'Extension 2'.

Part 1: Unity, proof and language

1. The unity of mathematics

Mathematics is an amazingly unified discipline, in which everything relates to everything else. Each of our present three courses incorporates this unity very well, with calculus as its centre, and the associated geometry and algebra tightly bound into the whole structure. For example, a 4 Unit HSC geometry question can often be answered using Euclidean, coordinate or complex number approaches, and a 2 Unit HSC question may approach an inequation through algebra, or through calculus and the geometry of a graph. More generally, the three courses are unified by their systematic treatment of degree 2 phenomena — quadratics, first and second derivatives, circles and conics, and complex numbers.

The Drafts, in contrast, are structured by 'Strands', which have no mathematical justification. They have appeared as dictated by ACARA, and seem to originate from Victoria, with its non-geometric traditions. Mathematics does not exists in 'Strands', and the adherence to them has split the Draft courses up into fragments that are poorly related to each other. From the 3 Unit Draft page 16:

- Why is 'Trigonometric Functions' a 'Strand', but exponential functions is not a 'Strand'?
- Why is 'Trigonometric Functions' a 'Strand' when it is part of 'Functions' and of 'Calculus'?
- Why is geometry not a 'Strand'?
- Why is algebra not a 'Strand'?
- Why is 'Functions' a 'Strand', when its equations would make it part of algebra, its graphs would make it part of geometry, and the study of its tangents and areas would make it part of calculus?

Most troubling of all is the ridiculous assertion that the 'Strands' are 'merged together' by modelling. Their material is unified by mathematics, not by models. Dividing material in this arbitrary way confuses everything, as can be seen from the structure of the three Drafts.

Matrices and linear algebra were developed to handle geometry, and then became a vital part of calculus. Statistics develops from calculus (the continuous distributions) and algebra (the discrete distributions) together. But these two suggested new topics have not been properly related to geometry, calculus or algebra as I have detailed below — they have been set adrift in the Drafts and confuse the structure of all three Draft courses.

2. Proof, understanding and problem-solving

Proof and understanding are two sides of the one coin, and are usually tested by problem-solving questions. This three-sided endeavour is the single goal of all mathematics, and all three courses should be doing everything possible to develop understanding and the skills involved in proof. The present courses, supported by the HSC examinations, do this superbly, and every student completing them develops a good idea of what proof and understanding mean in mathematics.

The demands for proof increase through the three present courses, so that students are challenged within their capacities in each course. The present courses are usually very clear about whether it is intended that students reproduce a proof of the result — usually tested by requiring a proof of a similar result or the solving of a similar problem — or whether they should be presented with a proof that is not examinable — and it is well accepted that teachers of less able students may weaken the normal rigour of a proof in these situations. Because of these demands for proof, students who have completed these courses have a good experience of what it means to do mathematics.

The Drafts have abandoned this excellent tradition. Results are typically introduced into the text with 'use', referring to the unfortunate 'choose and use' doctrine that came into mathematics education a few decades ago, but which has generally been rejected in NSW. It is a wonderful feature of these three NSW courses that almost everything can be proven, apart from foundational results, mostly involving limits, where only a satisfactorily intuitive account is possible at school. In mathematics, there is no 'understanding' without 'proof', and there is no reason for the Drafts to have abandoned this excellent and so carefully worked out part of the NSW tradition.

The most striking effect of this 'fear of proof' in the Drafts is the consigning only to the 4 Unit course of any discussion of proof. But proof and understanding are the same thing, and students in the 2 Unit and 3 Unit courses must be explicitly engaged with proof, at a level consistent with their abilities, otherwise the most basic method of the discipline would be withheld from them, and the courses could really not be described as mathematics courses. They would certainly not be satisfactory preparation for university study.

The two suggested new topics, in the form in which the Drafts present them, totally contradict the NSW tradition of understanding. Matrix multiplication is introduced as an arbitrary procedure, with no geometric discussion of the composition of linear transformations that motivates it, and the damage is compounded by a discussion of determinants that fails to give any means of evaluating other than 2×2 determinants. Ever more complicated ideas in statistics are introduced, but standard deviation, which is essential for everything that follows, seems to be undefined at the very start, so that everything in these sections is only blind use of unproven, arbitrary formulae. Their presence in the Drafts make the whole course incoherent as a course in mathematics.

3. Language

Language must always be used with total precision in mathematics — one misplaced word can destroy a mathematical sentence. Yet confused language is used throughout these Drafts, in turn indicating confused understanding of the mathematics by the writers, together with the failure to think through how material will actually work in the classroom.

From the content, I will take one extended example for many, because I know that others are putting together very lengthy 'catalogues of errors'. The third section of the 2 Unit course is called 'Working with Functions'. In the discussion of the 'equations of graphs of circles' (which should be just 'equations of circles'), the last dotpoint of page 35, we read:

> 'determine that for the relation to become a function, the domain must be restricted'

- It is the range of the relation that must be restricted. The writer is probably confused here by the construction of inverse functions, which are not being discussed.
- It is not 'the relation' that becomes a function, but a new relation that is formed by restricting the range of the old relation.
- There is no mention of how much the range is to be restricted, and whether it is implied that the curve still be continuous, or the domain remain unchanged. Neither is there any mention that with these two conditions (and using the equation in the next line), only two functions can be produced, namely $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$, and that they are semicircles.
- The word 'relation' has not been defined, and so cannot be used in the sentence quoted above — it is also missing from the Glossary. Thus, in turn, the concepts of the domain and range of a relation are also not available.
- The term 'relationship' is used twice in a different sense in the term 'inverse relationship' in the next section (page 36). The word 'linear relationship' is used on page 34 (7th-last

line) to mean something else again, but its use here is wrong — the verbs 'model' and 'analyse' require 'linear function', and the third verb 'solve' requires 'linear equation'. The word 'relationship' is also missing from the Glossary.

- The stem line above the item quoted above asks students to 'determine the equation of graphs of circles' (which should read 'determine the equations of circles'), but omits the necessary reference to Pythagoras' theorem, let alone its converse. Such constant denial of geometry is typical of the Drafts.
- Does the last item on page 35, 'find the equation of a circle given sufficient information', include the situation where three points on it are given, or where the centre and a tangent are given, or where three of its tangents are given? This is only a 2 unit course, and it contains no geometry. Such failure to explain what is required from students is unfortunately typical of the Drafts.
- Earlier in the same section (page 34), there is an 'Introduction to functions', yet the word 'function' has already been used extensively in the previous section on 'Trigonometric functions and graphs'
- That earlier section introduces ideas of transformational shifts, but these are not discussed in the present section, even though the equations of circles is the perfect place to discuss translational shifts.

The writers of this language do not understand how to write mathematics, nor how the parts of a syllabus fit together. Nor do they understand how a mathematics classroom works. They also have no respect for the care and wisdom and success of the present NSW syllabuses. These errors seem to be the result of cutting and pasting snippets from documents from other sources, with little concern to incorporate them into a coherent and teachable course in mathematics.

4. The Glossary

The Glossary is an exhibition of poor mathematical language and understanding, with problems in the majority of its entries. Again, others are compiling 'Catalogues of Errors', and I will mention only a few examples from the 2 Unit Draft to illustrate how far these Drafts are from mathematics.

- The omissions are many and surprising — 'regression' and 'correlation', for example. And 'set' on which everything depends. Also missing anywhere are an account of the logical words 'and' and 'or', which cause so much difficulty in compound conditions and in set notation, in particular because the word 'or' always means 'and/or' in mathematics.
- There are words, such as 'codomain' in FUNCTION, that are not mentioned in the Content.
- There are non-standard terms and notations (in NSW, perhaps not in Victoria):
  - 'line segment (interval)' rather than 'interval',
  - 'tangent line' rather than 'tangent',
  - $\leftrightarrow$ rather than $\Leftrightarrow$ for 'if and only if' (but words are always preferable for logic),
  - German fractur $\Re$ used for the set of real numbers,
  - Pascal's triangle drawn as an equilateral triangle, not as a right-angled triangular table so that its values can be read off from n and r.

  These seem to be the result of unedited cut-and-pastes from Victoria or ACARA.
- In many entries it is unclear whether sentences are definitions, or theorems, or explanations. All three approaches are needed, depending on the entry, but the Glossary should make it clear each time what is intended, because if a definition is confused with a theorem, then serious mathematics is impossible.

NON-INVERTIBLE MATRIX: 'A matrix [presumably A] is non-invertible if $\det A = 0$. A non-invertible matrix does not have a multiplicative inverse.'

Probably the second sentence is intended as the definition and the first sentence is intended as a theorem, which is confusing because it is contrary to the expected order. The qualification 'square' is omitted from both sentences, and the first is problematic for four other reasons:
- — The word 'determinant' is missing from the Glossary.
- — As remarked above, only the determinant of a 2×2 matrix has been defined (page 30).
- — Defining the determinant of a matrix larger than 2×2, and proving its invariance under the necessary row and column operations, is notoriously tricky and not possible in school.
- — Even then the theorem is too difficult to prove in school.

CONCAVITY: 'A function can be described as concave up or concave down.

'If it is concave up, then the second derivative is positive, and the rate of change (slope) of the graph is increasing.

'If it is concave down, then the second derivative is negative and the rate of change (slope) of the graph is decreasing.

'The concavity of a graph changes at a point of inflexion.'

The first sentence seems to be explanation rather than definition or theorem. It is wrong because a straight line cannot be described as either. It is also wrong because $y = x^3$ is concave up in some places and concave down elsewhere, so the property applies not to the function, but to parts of it. The second and third sentences require this same 'at a point' qualification.

Students routinely ask how to describe $y = x^4$ at the origin.
- — If concavity is defined in terms of chords, then the curve is concave up at the origin.
- — If concavity is defined by the sign of the second derivative, then the curve is neither concave up nor concave down at the origin.

If the second sentence was intended as a theorem, then it may be false, depending on which definition of 'concavity' is intended. But the second sentence was probably intended as a definition of concavity, and was incorrectly written backwards, or 'if' was confused with 'if and only if' — that would make the curve $y = x^4$ neither concave up nor down at the origin, and put it into agreement with the present syllabus, which is very clear at this point.

Is the fourth and final sentence a definition, or a theorem, or an explanation, of inflexions?
- — The entry POINT OF INFLEXION implies we should take the last sentence as a definition.
- — But neither is a correct definition because nothing is said about differentiability.
- — Vertical inflexions, as in $y = x^{\frac{1}{3}}$, have not been addressed.

A teacher dare not enter a classroom of able students without having considered such matters carefully, yet the writers seem not to have considered them.

- There are entries that do not address the issue:

NOMINAL DATA: 'Data that is listed by name (categorical) in which the order of the categories does not matter'. [The distinction between categorical and nominal has not been made.]

CONTINUITY: 'A function is continuous when its graph is a single unbroken curve'.
- — What is needed is not the continuity of a function as a whole, but the continuity of a function at a value of x. Did the writers read the present syllabus, which so carefully discusses the 'limit from above', the 'limit from below' and the 'value at the point', and requires them all to be equal for continuity at the value of x?

— Continuity of a function as a whole is tricky. Is $y = 1/x$ continuous, and is $y = x^2$ where $x > 0$ continuous — both are continuous at all points in their domains? Is $y = x^2$ where $1 \le x \le 2$ continuous — it is continuous at both its endpoints, if you have decided to define such a thing.

- There are constant serious confusions in the use of mathematical language. Some sentences are meaningless:

  ABSOLUTE VALUE: 'The absolute value or modulus $|x|$ of a real number x is the non-negative value of x without regard to its sign'. ['Non-negative value' is meaningless.]

  LIMITING SUM: 'The limiting sum of a geometric series, is the sum of the terms in the geometric sequence as it approaches its limit (the number of terms approaches infinity).' [Various ideas have been confused here, resulting in a meaningless sentence.]

- . . . and many are wrong:

  'A GEOMETRIC SERIES is a sum of the terms of a geometric sequence'. [Which terms? The usual approach is to define a series as the sequence of partial sums.]

  'A DISCRETE VARIABLE is a variable that has a finite number of possible values.' [A variable that can take any whole number as its value is discrete.]

  'A SCALAR is a constant that represents magnitude.' [Time is the most common scalar and is rarely a constant. 'Magnitude' is not in the Glossary and doesn't explain itself,]

- I will omit examples of the inelegant style and faulty syntax that is evident throughout.

Again, no one who understands mathematics would write these sentences (apart from the odd lapse or typo), and neither would anyone who understood the cleverness of the present NSW syllabuses, or had responded to the searching and brilliant questions of senior students in mathematics classes. These Drafts have been prepared by cut-and-paste, with little thought either for the mathematics or for the classroom, and should be discarded as any basis for the development of senior calculus syllabuses. Calculus must be written by a mathematician.

More generally, across all disciplines it is the responsibility of the Board to show leadership in using language elegantly and correctly. Teachers are continuously setting standards and giving examples with every word they say and sentence they write. The Board, whose standards must be higher than an individual teacher's, cannot issue documents with such poor language.

5. The language of the present syllabuses, and of the Drafts

The Draft's unsatisfactory Glossary notes on continuity, mentioned above, should be contrasted with the present syllabus' extended exposition of continuity covering two thirds of page 50. This is the sort of clarity that teachers rightly expect from a syllabus when a difficult concept is introduced.

Some of the writers' difficulties in the Content sections may be the result of constraints forced onto them by the Board, in that each part of each 'content' of each section is written as one huge ugly sentence (with no concluding full stop). Each new dotpoint or dashpoint then begins with a transitive verb, seemingly drawn from a fixed list of stem verbs, with the common subject 'Students' way up above. Frequent mock-legal language adds to clumsiness. I have consistently found the resulting text insanely difficult to read, and so often when I look closely, I genuinely cannot work out with any certainty what is being said.

Mathematics has a very long tradition from ancient Greek times of using very precise, straightforward language that is different from other disciplines. When we teach mathematics, we must also teach how to use its language. Forcing the language of educational theory onto mathematics, as is done here, thus stands in contradiction to the task of teaching mathematics, and it is no wonder that so much of the content is confused, fragmented and wrong.

The word 'prove', together with its near synonyms, is the most important word in mathematical prose, because it announces that a mathematical task is to be carried out. Yet the word 'prove' has clearly been removed from the allowed list of stem verbs because it never occurs as a stem verb throughout the 2 Unit Draft. Interestingly, the writers seem often to be using the words 'establish' and 'determine' as synonyms for 'prove', but even this is unclear because both these verbs are also used in various other senses, and 'determine' is often misused.

It would perhaps be interesting to ask why educational theorists believe that mathematics must be written in this turgid style in which meanings cannot be clearly expressed nor distinctions made, or why they believe that 'prove' should be removed from mathematics. But the immediate question is: 'Why is the Board enforcing such nonsense on the writers, and thus sabotaging its own syllabuses?'

6. The 'Objectives' of the present courses

The Board once wrote an excellent statement of the intentions of the present 2/3 Unit courses at the start of that syllabus (*Mathematics 2/3 Unit Syllabus — Years 11–12*, pages 7–8). The Drafts manifestly fail to meet these high standards. In particular, I have copied verbatim below the 'Objectives' of the present 3 Unit course. Their clear and balanced statements of the course's objectives, and how they can be achieved, can be read now as the Board's own rather savage criticism of these Drafts:

> Specific objectives of the [3 Unit] course are:
>
> (a) to give an understanding of important mathematical ideas such as variable, function, limit, etc, and to introduce students to mathematical techniques which are relevant to the real world;
>
> (b) to understand the need to prove results, to appreciate the role of deductive reasoning in establishing such proofs, and to develop the ability to construct these proofs;
>
> (c) to enhance those mathematical skills required for further studies in mathematics, the physical sciences and the technological sciences.
>
> For achievement of these aims, the following points are important:
>
> (i) Understanding of the basic ideas and precise use of language must be emphasised;
>
> (ii) A clear distinction must be made between results which are proved, and results which are merely stated or made plausible;
>
> (iii) Where proofs are given, they should be carefully developed, with emphasis on the deductive processes used;
>
> (iv) Attaining competence in mathematical skills and techniques requires many examples, given as teaching illustrations and as exercises to be undertaken independently by the student;
>
> (v) Since the course is to be useful for concurrent studies of science, industrial arts and commerce students could be given some experience in applying mathematics to problems drawn from such areas. Realistic problems should follow the attainment of skills, and techniques of problem solving should be continually developed.

Part 2: Syllabus content

1. Geometry

Geometry and arithmetic have been and remain the two contrasting intuitive bases of mathematics. Arithmetic begins with counting and rhythms and soon includes algebra, and geometry begins with what we see. The two intuitions have been axiomatised and unified in many ways, most famously by Euclid in his *Elements*. Then Descartes' systematic use of the coordinate plane unified the studies of geometric curves and algebraic equations, enabling the coordinate plane later to became the place where the new ideas of calculus could be represented.

Our NSW courses have followed this development, where on a geometric–algebraic basis we build successively trigonometry and coordinate geometry up to Year 10, leading in the senior years to calculus and further development of geometry and algebra. It is a wonderful tradition, because students tend to find algebra difficult to assimilate, whereas they understand very quickly when diagrams and geometric representations are used. The diagrams stimulate their imagination and enable them to grasp a theorem or a problem as a single idea. The Drafts have abandoned this tradition — Euclidean geometry has been consigned to the 4 Unit course, and some scraps of coordinate geometry are consigned to the content of M-A1 *Working with functions*.

Thus the 2 Unit and 3 Unit courses are missing the following foundational ideas from Euclidean and coordinate geometry. Nearly all 2 Unit students, and many 3 Unit students, need further work on these things to consolidate ideas whose significance they did not have the maturity to understand in Year 10:

- Similarity is needed for the definitions of the trigonometric functions, which are ratios of lengths, and congruence is needed for the application of trigonometry to right-angled and non-right-angled triangles. In particular, it clears up the ambiguous case with the sine rule.
- Pythagoras' theorem and its converse, plus the theorem that a tangent of a circle and the radius at the point of contact are perpendicular, are needed to handle the cartesian equations of circles, distance problems in the coordinate plane, and the geometry of tangents and normals used in introductory calculus.
- Congruence and similarity are needed for the transformations of graphs.
- A systematic treatment of coordinate geometry is required, as in the present 2/3 Unit courses, as a preparation for calculus that takes place in that coordinate plane.
- Area formulae are required for integration.

Calculus can then develop with diagrams at every stage of the work, and in particular with curve-sketching playing a central motivating and organising role. Students' love of calculus in NSW is due in large part to the fact that they can see it happening in the graphs of the functions that they are dealing with.

Euclidean geometry plays another role in the NSW syllabuses as the place where students learn what proof is all about. Proofs that involve only words and algebraic symbols can be difficult to assimilate, and students can easily lose track of what is being assumed and what is being proven. But when there is a diagram, the logical progression of the ideas becomes clear. Euclidean geometry was re-introduced into the NSW senior courses some decades ago, not only because of its fundamental importance in mathematics, but because its visual proofs are the perfect introduction to proof, which is the absolute basis of all mathematics. The writers are misjudging our students very badly, and selling them short, in denying that anyone except a 4 unit student can cope with proof.

2. Review of earlier work

It is not appropriate for the Board to omit many essential introductory topics from its senior courses.

- Most 2 Unit students need quite a bit of review, at a higher and more general level, of much Year 9–10 work, particularly Euclidean geometry, coordinate geometry, quadratics, trigonometry and indices, if they are to succeed in calculus.
- Some of these topics will not have been taught to some students in Years 9–10, depending on the school and the students' previous mathematics course.
- If these essential introductory topics are omitted from the HSC courses, then pressure of time will cause them not be taught well in Years 9–10 in some schools.
- The weakest 2 unit students traditionally get the chance here to show their mastery at least of the foundations of the course and gain some marks.

All the topics should be in the senior courses. Each can be presented in a more elegant and unified manner than was possible in Year 10, and related more strongly to other topics. Teachers should have discretion as to how much time each class and student needs to spend on them.

Completing the square is reviewed algebraically in the second topic (page 34, A1.1), but is not mentioned in the sketching of a quadratic function later in the same topic (page 35, first dash point), nor is completing the square related to the transformations of the previous section. Again, the writers are unconcerned with relationships between algebra and geometry.

3. The beginnings of calculus in the 2 Unit course

The sustained and systematic treatment of calculus is the greatest strength of the present courses, but unfortunately the Drafts have discarded the very things that have contributed to these strengths.

CALCULUS, GEOMETRY, RATES AND MOTION: The present course begins calculus by introducing the derivative as the gradient of the tangent, and from this it develops the definition of the derivative as a limit. This is difficult and demanding work, but is well based on the geometry of tangents and the coordinate geometry of lines, and it is generally met with enthusiasm by the students. The Draft confuses this development by talking also about rates in general, and then introducing as well the particular case of motion.

The Drafts, consistent with their constant denial of geometry, introduce the derivative as a rate, which, because it involves time, is more an applied mathematics notion. This makes it far harder to gain a clear picture of what the derivative is. Students in Year 12, particular in 2/3 Unit, find it surprisingly difficult to move easily between x and t on the horizontal axis, and will be particularly confused when they are just learning to differentiate. Yet the writers want Year 11 students to assimilate the relationship between rates and tangents, at the very same time that they are trying to assimilate the new and challenging idea of the derivative as a limit.

Not only does this approach muck up the presentation of the derivative, but it also mucks up the exposition of rates and motion, which need their own sustained treatment after students already have the derivative and the integral. The present NSW syllabuses have been very successful, both in introducing calculus, and in developing its applications. All that experience has been ignored in the Drafts.

CALCULUS NEEDS A SUSTAINED TREATMENT: Calculus completely transforms a student's view of mathematics. The present courses follow the introduction of the derivative with geometric problems, and then immediately pass to the chain, product and quotient rules, to the derivative of x^n for all real numbers n, and to second and higher derivatives. This sustained attention to the derivative, using only algebraic functions, allows the transformed view of the subject to take place. It also strengthens students' algebraic skills, and allows a large number of interesting functions to be analysed. The Drafts, however, consign these three rules and the second derivative to Year 12 — and I cannot make out from page 42 the values of n for which the derivative of x^n is to be found, nor what methods of proof are available.

CALCULUS AND PROOF: The long tradition of proving the various formulae in calculus, as appropriate for the skills of each class and student, has been abandoned. For example, the chain, product and quotient rules, and the derivative of x^n, receive only the usual verbs 'use', 'apply' and 'understand'. This changes mathematics from a sequence of logical arguments to a set of arbitrary rules.

4. Calculus, curve-sketching and the special functions

The present courses have a wonderfully logical approach to these things:

- Initially, curve-sketching of various algebraic functions is developed alongside the sustained idea of a function, allowing such things as domain, range, zeroes, asymptotes and continuity to be discussed. The more general idea of a relation is introduced, along with the vertical and horizontal line tests, inequations and regions.
- Then calculus is developed for algebraic functions, using the geometry of tangents to motivate the limit, and the chain, product and quotient rules are proven, together with the derivative of x^n for all real values of n.
- Using at first only these algebraic functions:
  — The geometry of stationary points and inflexions is developed.
  — This geometry is applied to curve-sketching, which now involves the analysis of a function in terms of symmetry, zeroes, asymptotes, stationary points and inflexions.
  — This geometry is applied to maximisation problems.
  — Much attention is given to the constant of integration when finding an anti-differentive.
  — Integration is developed using the geometry of areas to define the definite integral, and using sandwiching of areas to develop the fundamental theorem of calculus.
- EXPONENTIAL AND LOGARITHMIC FUNCTIONS: These are introduced with whole number bases such as 2, 3 and 10, and they are graphed and studied as functions. Then the question of gradients is introduced, and with varying degrees of rigour, it is proven, first, that the derivative of an exponential function is a multiple of itself, and secondly, that there is a particular real number $e \doteqdot 2.7$, often called *Euler's number*, for which the exponential function e^x is equal to its own derivative. Alternatively, and a little more rigorously, one may begin with the search for a primitive of $1/x$. This, and the change-of-base formula, allows all the previous methods of calculus to be applied to exponential and logarithmic functions, and the whole structure of calculus as previously introduced is reviewed with these new functions added.
- TRIGONOMETRIC FUNCTIONS: These are introduced, still in degrees, and graphed and studied as functions. When calculus is introduced, it becomes clear that angle size can be redefined so that $\sin x$ has derivative exactly $\cos x$ rather than a multiple of it. This is radian measure, based on the other famous transcendental number π using geometric ideas of similarity again. All the previous calculus is now extended once again to include the trigonometric functions.
- APPLICATIONS: These are studied only now, when all the methods are at hand, but it is systematic. There is a sustained treatment of motion, and a sustained treatment of rates, which includes some financial mathematics.

This structure works wonderfully well, because ideas are constantly revisited and developed, and students can see calculus gradually expanding to take in almost all their previous mathematics. Large numbers of students develop an exhilarating sense of mastery at having assimilated such complex new ideas and being able to answer so straightforwardly such seemingly difficult problems. There are a couple of minor problems:

- Transformations of these very symmetric curves are not handled systematically enough. This work should be done before calculus, establishing clear equivalences between the geometric and algebraic representations of each transformation, then continued as calculus develops.
- The work on rates does not satisfactorily cover the simpler work when a quantity is differentiated to find the rate. This is easier, and make the later integral of rates more understandable.

Contrast the disjointed, confusing presentations of the Drafts. Some examples amongst many:

- Radian measure is introduced in the very first Year 11 topic 'Trigonometry and Radians' (page 30), when it is completely unmotivated by the structures of calculus. Once again, mathematics presented in this way appears arbitrary and unmotivated.

  Similarly, the base e is introduced into the fourth section on exponential and logarithmic functions, again completely unmotivated by the structures of calculus, and again giving the impression that mathematics is arbitrary and unmotivated.
- The word 'functions' is used in the second section (page 32) on 'Trigonometric Functions and Graphs', yet functions are only formally introduced in the third section (page 34).

  Transformations of trigonometric graphs, and the associated algebra, is introduced in the second section, but not mentioned again in the third section on functions (page 34).

  Conversely, the associated function terminology of domain and range introduced in this third section is never applied where it is really needed, that is, to the trigonometric, exponential and logarithmic functions of the second and fourth sections.
- The concepts of even and odd functions, and more generally the need to observe possible symmetries in any function studied, are very important throughout the course, but despite the mention of transformations, they have been omitted or forgotten. This is particularly the case with the multiple symmetries of the trigonometric functions, where the ASTC identities are so clearly visible on the graphs. Again, the algebra–geometry connections are missing.
- Transformations of graphs do recur as the third topic 'Graphing Techniques' in Year 12 (page 54), but without mention of the trigonometric functions discussed at the start of Year 11.

  The same section, last dotpoint, mentions stationary points, which are only introduced later, in the fourth topic on page 57. It also appears to imply that there will always be symmetry about a stationary point.
- The fourth topic, 'Differential Calculus' (page 56), wants graphing software to establish that e^x is its own derivative.

  Similarly, the fifth topic 'Integration' wants graphing software to justify the fundamental theorem of calculus (page 59), which, by the way, has lost the name.
- There is insufficient emphasis on systematic curve sketching once the geometry of the derivative has been completed.
- The extended trapezoidal rule (page 58) should not be there, because the point of the trapezoidal rule in the courses is not to perform tedious calculations, but to provide geometric intuition about the meaning of the definite integral, and to give a simple approximation for which it is usually known, by concavity, whether it is greater than or less than the exact

value. Discussion of such inequalities, however, has been omitted. (But it was good to omit Simpson's rule because the geometric meaning is too obscure for school.)

5. Disjointed applications of calculus

The worst result of introducing the derivative as a rate is that the Draft now lacks any systematic exposition of applications of calculus, that is, of motion and rates of change.

Most disastrously, integration now comes *after* all the applications have been discussed, so that integrating acceleration to get the velocity, or velocity to get the displacement, or a rate to get the quantity, seem to be missing from the course. Learning how to evaluate the arbitrary constant of integration now goes nowhere.

Particularly damaging is the loss of any coherent exposition of motion. This is the one application where both the first and second derivatives can be modelled respectively by velocity, which you see, and acceleration, which you feel. A single problem can involve two successive constants of integration, so that the significance of boundary conditions becomes clear, even though the student has not encountered the complexities of differential equations.

FINANCIAL MATHEMATICS:

- Financial mathematics is the perfect place to compare continuous compounding and discrete compounding. For example, if prices are increasing at 4% per annum, one could use discrete compounding to find the price after 5 years, then use exponential growth to find the price after 6 years and 5 months. This very significant connection between GPs and exponential functions is otherwise not made.
- Housing loans and superannuation with constant interest and constant deposits have been found by teachers to be too hard for 2 Unit students, but the argument for including them is very strong, in that most students will have to think about them in a decade or less, and they are an excellent application of GPs. Those two examples, however, are quite enough, and it would be most unwise to change to a new idea of 'annuities' which may combine deposits with withdrawals. By the way, the definition (if it is intended as such) of ANNUITY in the Glossary is unclear and seems to differ from the use of the word on page 61.

6. Matrices in the 2 Unit course

- Matrix multiplication is motivated by the composition of linear transformations. Without that understanding, matrix multiplication is just an arbitrary, pointless rule. A matrix should certainly not be introduced as a place 'for storing and displaying information' (page 50). But of course the writers are avoiding geometry at all costs.
- As I have said above, 'determinant' is far too hard a concept for school. Page 50 defines a 2×2 determinant as a special case and then asks for a 3×3 determinant to be evaluated.
- Cramer's rule (page 67) is a distraction that is not worth teaching. Matrix equations should be handled by echelon form (or possibly by eigenvalues).
- The word 'vector' is defined neither here nor in the Glossary, and typical of these Drafts, no mention is made of its straightforward geometric meaning in n-dimensional coordinate space.
- Correspondingly, there is no motivation to solve a 'matrix equation' (page 70).
- From the point of view of a 2 Unit class, the 'applications' are obscure and difficult. It is ludicrous to be teaching such things when the basic geometry of matrices is missing.
- Most importantly, no attempt whatsoever has been made to relate this material to the rest of the course. This is contrary to mathematics, and contrary to the present 2 Unit course.

These two sections on matrices have no place in the 2 Unit or 3 Unit mathematics syllabuses, and given the difficulty of the concepts, cannot be rewritten satisfactorily so that they do fit. One could perhaps argue for the inclusion of matrices in the 4 Unit course, where it could be developed from a proper discussion of linear transformations. But my experiences of trying to teach exactly that to my most able classes have convinced me that apart from 2×2 matrices, which achieve little, it would be very difficult to explain and to relate to other topics. There are better things to teach to very able students, such as the present 4 Unit course.

7. The 3 Unit course is inadequate for university mathematics

The problems of these Drafts originate in the 2 Unit course, as I have explained, so I will comment far more briefly on the 3 Unit and 4 Unit courses.

The 3 Unit course builds on the 2 Unit course, and correspondingly the 2 Unit Draft's problems of incoherence, lack of understanding and proof, denial of Euclidean and coordinate geometry, and inadequate preparation in calculus, are carried over to the 3 Unit Draft.

The most significant problem with the 3 Unit course is that it no longer provides an adequate preparation for university mathematics. This has always been regarded as the main goal of the 3 Unit course, as can be seen in the course objectives that I quoted on page 7 above, with the 4 Unit course added to challenge very gifted and motivated students. These topics from the present syllabus are no longer in the course:

- Proof by mathematical induction is missing. This alone will cause huge difficulties at university. Every course there will be held up by the need to explain this ubiquitous method of proof.
- In fact, proofs of all types have been consigned to the 4 Unit course, so that 3 Unit students will not be taught the fundamental method of the discipline.
- Along with proofs, Euclidean geometry, which was omitted from 2 Unit, is not even included in 3 Unit. Once again, all the imaginative powers of geometry are missing.
- Geometry is a far better preparation for matrices and linear algebra than ignoring geometry and teaching by rote the multiplication of matrices and the evaluation of determinants.
- Three-dimensional trigonometry has been omitted, presumably because students can't understand three-dimensional geometry. Thus the course is no preparation for engineers, architects and geologists, let alone people taking physics or studying complicated organic molecules.
- Parameters are missing. Interestingly, the word 'parameter' is thrown about in the 3 Unit Draft and appears in its Glossary. Parameters provide a perfect example of the chain rule in action, and they give an insight into how time is used in physics. They are very easily explained in school, and then they cause no trouble in university.
- Locus is also missing. This is an old idea, motivating many curves in geometry, and is very important in much abstract mathematics, physics and engineering.
- Projectiles use most of the calculus, algebra and trigonometry in the 3 Unit course, and are a wonderful application, historically justified, of what is now possible using the the calculus taught so far. They also naturally involve parametric equations.
- Motion with $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$ is missing, for no good reason. It is an excellent way of solving physical problems (and we know that it needs suitable prompting in an examination situation).
- There is no coherent treatment of rates of change, despite all the chatter, and no coherent treatment of the relationship between discrete and continuous rates (which is only covered by implication even in the present course).

- Simple harmonic motion is missing. This means that the whole point of studying the calculus of the trigonometric functions is missing, because SHM is the physical model that scientists use for all other sorts of periodic motion. It is also the cornerstone of Fourier analysis, which is so important in so many university courses. But SHM should be defined by $x = a\cos(nt+\varepsilon)$, because the second order DE $\ddot{x} = -n^2x$ has created too many conceptual problems at school.

What we have at present is a totally unified and demanding 2/3 Unit course that prepares students well for university mathematics. In contrast, the Draft is a disjointed mess of bits and pieces that will leave students, and their teachers, confused as to what it is all about. I have no idea how the universities would pick up the pieces if this course were introduced.

8. The 4 Unit course — Dynamics

It is a mistake to introduce dynamics, rather than kinematics, into school mathematics. In the present course, kinematics is there to model calculus and show some extraordinary things that can be achieved with it. It is very difficult to explain what a 'force' is in a mathematics classroom. Is it to be *defined* by $F = m\ddot{x}$? Or is force to be defined in some other unspecified way, after which this 'law of physics' will be taught as a fact without any experimental confirmation? Or are teachers to create experiments to demonstrate it? Any of these three ways is unsatisfactory.

9. Conservation of momentum

There is no mention of conservation of energy, which is a really basic and important law of physics and all of science, and should surely be discussed before conservation of momentum is mentioned. Thus the material is also incoherent as physics.

Momentum is manifestly *not* conserved when one slides a book along the floor by giving it a push, unless you talk about completely unobservable processes at work in the individual molecules of the book and the floor. It is not like conservation of energy, where kinetic energy is transformed into heat energy that you can feel and measure.

Who knows what 'exists' means when it comes to concepts such as energy and momentum, but momentum really does carry the impression of being a 'physicist's construct' rather than a real existing thing.

Better to leave physics to the scientists, and mathematics to us. Kinematics is a wonderful topic, and extends 4 Unit students as far as they can go, as the recent HSC examination papers show.

10. Vectors in the 4 unit course

Vector proofs of geometric results are a dubious exercise at school. These approaches are too esoteric for even 4 Unit students to work confidently with them and develop their own proofs.

The sections on vectors do not go very far, they will complicate rather than illuminate the succeeding kinematics, where coordinate vectors are all that is needed, and the rather advanced calculus topics of the present 4 Unit course are much more interesting and exciting. They and the physics seem to be fragmenting what is at present a wonderful and inspiring course.

11. The exponential form of a complex number

The section on complex numbers requires students to write a complex number as $re^{i\theta}$. This is yet another object that can neither be proven nor explained. It is only with the insights of calculus with complex numbers, and the convergence of power series, that this expression makes sense. The ideas of complex numbers are hard enough without bringing in unexplained notation such as this.

Universities, for their own reasons, sometimes use this form of a complex numbers before it is explained because it makes various operations very compact. If we use it at school, it makes it easier for them. But it undermines badly the logical coherence of our school course because we cannot justify it, nor explain why using it gets the right answers. Schools' best service to the universities is not to teach such unexplained notations, but to prepare students with a genuine and unified experience of mathematics and an enthusiasm for it.

12. 'Harder questions on . . . '

Earlier versions of our current Extension 1 and Extension 2 syllabuses had sections entitled 'Harder questions on the 2 Unit course' and 'Harder questions on the 3 Unit course' respectively. This invited extension students to see all their senior mathematics as a unity, and it enabled the examiners to develop some very difficult implications of the foundations of calculus in the 2 unit course. Traditionally, the hardest questions on the Extension 2 paper have been of that type.

The present syllabuses unfortunately omitted this section, but the unified nature of the courses has meant that the examiners hardly needed to change their practices.

The Draft Extension 1 and Extension 2 courses are so fragmented that it is unlikely that examiners would be able to continue this tradition. I would recommend that the sections be restored in any new Board Extension syllabuses so that the material is explicit, and so that, for Extension students, the totality of their mathematics courses is better unified.

13. The order of the topics

I notice that extension students may be taught the less demanding course 'prior or concurrently', which is good. The present arrangements, however, are far weaker, and in particular, one may teach both years of the 2 Unit course before embarking on the 3 Unit course.

The Board knows better than the rest of us how many different situations there are in schools, whose teaching of the extensions courses is so often complicated by small numbers of students, by the availability of qualified teachers, by acceleration programmes, and by students' experiences in Year 10 mathematics. It would be a great relief if the Board would remove these artificial barriers between the Preliminary and HSC courses, and allow schools to teach the material in whatever order best suits the situation in each school, with regard only to the systematic development of the mathematics. The Board should recommend the order, perhaps strongly, but not mandate it.

This in turn affects the Board's ban on examining Year 11 material in Year 12, which I have commented on in the Assessment section below on page 22.

14. Extension 2 students and the 2 Unit Year 12 course

On a related matter, the Extension 2 draft does not seem to make it clear enough that Extension 2 students need to study the Year 12 part of the 2 Unit course.

Part 3: Statistics

I began here with the intention of closely criticising the statistics sections, but after writing a couple of pages, it became clear that the statistics Drafts are so poor that it was impossible to carry this through. I therefore stopped, and simply summarised my conclusions about the rest of the statistics material. I have, however, left in place my notes up to 'Probability Distributions'.

I cannot follow, from the Content or the Glossary, what the Draft is asking teachers to teach. Discussion about including statistics in the senior courses has been going on now for about 30 years. If this is the best that can be done, then it is clear that statistics should be dropped completely.

1. Embedding statistics into the mathematics course
Proving things is how mathematics is done, yet though the words are longer and more terrifying, little is proven in the statistics sections. From the start, nearly everything seems to be 'choose and use'. This problem begins with the first section 'Descriptive statistics' (page 44), where the most basic concept of population standard deviation seems never to have been defined, either here or in the Glossary. It is a simple quadratic object, and these students have been studying quadratics with increasing intensity now for three years.

We teach 2/3/4 Unit students to initiate them into the methods of mathematics so that they have the option of taking further mathematics at university. When we teach them arbitrary unproven formulae at school, we are deceiving them about the nature of our subject and wasting their time.

Once the decision to avoid standard deviation was made, everything else in the statistics sections had either to be reduced to 'choose and use' with arbitrary unproven formula, or given only a qualitative account using words that will forever remain unclear to students. Both approaches are used, but neither leads to proof and understanding.

2. Section M-S1 Descriptive Statistics
There was no proper review of coordinate geometry, but here the review of Years 9–10 descriptive statistics goes on forever, with all sorts of pictorial representations of data that will bore the students, waste their time, and lead nowhere. This section should be omitted.

3. Section M-S2 Probability
This short section seems reasonable at first glance, but has many problems.

- There is no discussion of counting, with or without replacement, either at the start or later.
- There is no simple explanation of the distinction between experimental and theoretical probabilities, nor clarification that the probabilities in this section are theoretical.
- There is no mention of analysing the possible results of an experiment into a set of 'equally likely possible outcomes'. Such analysis is the key to everything in this and the next section.
- There is no definition of the very tricky word 'probability', either here or in the Glossary. What is the conceptual framework here? Is it lack of knowledge, or a statement about the future, or something else? What does 'equally likely' mean? What does '$P(E) = \frac{1}{6}$' mean?
- In the dashpoint 'determine relative frequency as probability', does 'determine' mean 'define', or 'estimate' or 'measure' or something else? Has the clause been reversed?
- The second dotpoint should give attention to the words 'and', 'or' and 'not' in the discussion of intersection, union and complement, particularly the fact that 'or' means 'and/or'.
- The third dotpoint has two formulae, but does not explain if or how they are to be proven.
- The fourth dotpoint introduces conditional probability, which seems very tricky for 2 Unit.

— What is the definition of conditional probability? Is it the formula in the fourth dotpoint $P(A \cap B) = P(A|B)P(B)$, or is it the verbal definition that I found in the Glossary? If it's the verbal definition, how is the formula to be proven?
— Defining independence by the formula rather than verbally seems too abstract for school mathematics. There needs to be a clear conceptual framework for these things.
— Where does the multiplication law in the fifth dotpoint come from? In the present courses, we prove it by taking the cartesian product of the sample spaces and then of the event spaces, but perhaps the intention here is that it come from the formula $P(A \cup B) = P(A|B)P(B)$.
— There needs to a clear exposition of the use of the multiplication rule to solve problems. Or perhaps such problems are not intended?
— The final clause is obscure.
— Tree diagrams are the visual representation of the multiplication rule, so tree diagrams should be in the fifth dotpoint rather than in the first. Why does the Glossary contain only counting-tree-diagrams when the only tree diagrams in the Content are probability-tree-diagrams?

4. Section M-S3 Probability Distributions
This section is unteachable:

- The first dotpoint leaves some major ideas unexplained.
  — What is a random variable? It is defined neither here nor in the Glossary.
  — What is a probability distribution? It is very ambitious to present such a difficult, abstract idea to 2 Unit students, but it is defined neither here nor in the Glossary.
  — What is the 'mean or expected value' of a discrete random variable? It is not defined here, and the Glossary entry is meaningless.
  — How can one 'evaluate the variance and standard deviation of a discrete random variable' when, it seems intentionally, neither term seems to have been defined?
- The term 'Bernoulli random variable' should be defined here, not just in the Glossary. The Glossary should explain why the entry BERNOULLI RANDOM VARIABLE talks about 0 and 1, whereas the entry BERNOULLI TRIAL talks about 'success' and 'fail'. This last entry is wrong because the key phrase 'exactly two' has been omitted. There are also spell-checker problems, in that 'labelled' rather than 'labeled' is the accepted Australian spelling, so perhaps the text was imported from the US.
- With the third dotpoint, a huge teaching problem emerges. The third dotpoint wants the binomial distribution taught in terms of the binomial expansion of $(x + y)^n$ to 2 Unit students in Year 11, when we know that 3 Unit students in Year 12 find the binomial expansion extremely difficult. This cannot be taught.
- I cannot follow the third dotpoint. It wants students to use the expansion of $(x + y)^n$, but it does not want the general term, and I don't know for what purpose it is to be used.
- Nor does it mention the Pascal triangle, which is the most straightforward way to handle any particular expansion for a low value of n. The Pascal triangle at least is visual and gives a clear and interesting representation of the coefficients — it would be poor practice to teach the expansion without the triangle.
- Yet the second last dashpoint talks about ${}^{n}\mathrm{C}_r$, which looks pretty general.
- How does one 'use' the variance once it is calculated by a presumably arbitrary formula?

5. Suspension of close criticism of the Draft statistics
There is no point continuing with these details. The sections present little understanding of how the mathematics hangs together, there is little idea what is going to be taught and how it is to be

taught, there is little care with logical progression or expository language, and there is no indication of knowledge of the content and reception of the present courses. Material from goodness knows where seems to have been cut and pasted into the Drafts without effective editing. It is unclear how 30 years of discussion has led to such a disaster, but it is clear that the statistics in these Drafts is so confused that it would be most unwise to work from them to any final document.

I will now start again and make a series of recommendations about the statistics. I have already recommended above that descriptive statistics be removed.

6. Probability and probability distributions

2 UNIT: The 2 Unit counting and probability material in the present syllabus is all good. I have always felt, however, that it could go a little further and present a very simple treatment of counting words (permutations with any number of repetitions), permutations and combinations. Such a treatment could avoid the binomial expansion altogether by defining ${}^n\mathrm{C}_r$ in the far more natural way as the number of r-subsets of an n-set.

Moving on to the binomial distribution seems unwise. First, probability distributions are conceptually too much, although one could perhaps succeed if it were prepared by many simple examples. Secondly, the necessary identities on the rows of the Pascal triangle are quite beyond 2 Unit, and this seems impossible to avoid.

3 UNIT: The present 3 Unit material on counting and probability is excellent and suitably demanding, with the one disadvantage that ${}^n\mathrm{C}_r$ is defined as the coefficient in the binomial expansion, which has proven a little too ambitious and confuses even able students. That should be a theorem, in a later topic after ${}^n\mathrm{C}_r$ has been introduced as above.

There is little purpose in proceeding to the binomial distribution, but it would be possible with these more able students. It would have to wait until after the binomial theorem and the necessary binomial identities had been developed, after which they would have the tools to prove the mean and variance formulae.

7. The normal distributions

There are two barriers here. The most serious is the idea of a continuous probability density function, the less serious is the lack of a primitive of $e^{-\frac{1}{2}x^2}$ amongst the functions available in the 2/3 Unit courses. The strong arguments for including it are its ubiquity in statistics, and its routine presence in the popular media. I would prefer to leave it out, reckoning that just as introducing e^x is motivated by calculus, so the normal distribution is motivated by the central limit theorem.

If it is to be included, however, it must be carefully explained as an application of integration, and presented as rigorously as reasonable. The presentation should include:

- The new ideas of probability density functions, integrals as probabilities, and expectation, all need careful development with much simpler functions.
- Curve-sketching techniques should be applied to $y = e^{-\frac{1}{2}x^2}$, with attention to the inflexions and the asymptotes, and the speed with which the function decays.
- Students need to told that its integral from $-\infty$ to ∞ is $\sqrt{2\pi}$. Later, some 4 Unit students will perhaps be able to prove this. After dividing by $\sqrt{2\pi}$, we have the required probability density function. Comparison with $y = \dfrac{1}{1+x^2}$ would now be useful, where the curve may look similar, and the integral is still finite, but decay is much slower.

- The mean is 0 by the symmetry of the integral. The variance calculation can be understood by 3 Unit students, but only by very able 2 unit students.
- The connection between the variance and the point of inflexion can be drawn out.
- The function is now embedded within calculus, and its distinctive statistical aspects and methods can now be introduced.

8. Correlation and regression
This material is too hard for school, and does not fit with the other mathematics in the courses. There is an unexplained mention of 'least squares', but doing anything properly with least squares is too difficult for school. It is not clear that, despite the fearsome language, anything mathematical is really gained beyond the box plots and discussions of Year 10. Causation is at the very centre of science, but it is for science lecturers to teach, in the context of scientific method and the statistics appropriate for the particular discipline. This whole topic should be removed from the courses.

9. Sampling and estimates in the 3 Unit course
There is nothing good to say here. Neither students nor their teachers will be able to cope with this in the classroom. It is way beyond school, and makes no sense to anyone until, in later studies, they are actually faced with the problems of constructing a survey or criticising results obtained from a survey — then it will make very good sense.

10. A general remark about statistics
There is widespread opinion that serious statistics does not belong in a school mathematics course, because not only can things not be proven and explained, but they also cannot be motivated. When a young person embarks on, for example, a university psychology course, she will be taught scientific method as practised in that discipline, and realise how difficult it is to say anything at all unless it is backed up by really complicated statistics. At that point, the student is motivated to learn statistics and to understand what all its myriad terms and ideas mean in the context of quite concrete situations to whose analysis she is committed. The school courses should be preparing the ground for statistics as firmly as possible by teaching algebra, calculus and geometry as extensively and as rigorously as is appropriate for each student. Mathematics is fiendishly difficult, and these topics are really the only branches of mathematics in which school students, who mostly have as yet no vocation, can make progress.

In contrast, statistics at school will come across to the students as a mass of disjointed material, unconnected with rest of the course. It will appear arbitrary and pointless. It will not teach them mathematics, but instead will confuse the structure of the courses and confuse the students.

Part 4: Assessment

The suggested assessment procedures are unfortunate in several ways. They will not assess mastery of the course properly, they will bore students and teachers, and they will not encourage students to challenge themselves appropriately. Also, the examination, which is taken very seriously in subsequence years and becomes *de facto* part of the syllabus, should be conveying the same values of rigorous imaginative thought as mathematics itself.

1. Common questions on two papers

THE 2 UNIT/3 UNIT HSC EXAMINATION OVERLAP:
There is a proposal for 25% of HSC marks to be common questions on the 2 Unit and Extension 1 papers. This would be a very bad decision for a number of reasons:

- There is no control of 'time on task' by students during the examination, and students of the more difficult 3 Unit paper would naturally spend less time checking the common questions, so any scaling based on these common question would be invalid.
- At present, many stronger 2 Unit students risk taking one extension unit because they know that 100 marks depend on their 2 Unit paper, and only the last 50 Marks depend on the often very difficult Extension 1 paper. Were 150 marks to depend on the one paper, such students would not take the risk. The effect would be fewer students taking the 3 Unit course and thus preparing themselves for university mathematics.
- An Extension course should be what it says it is. Students complete and are examined on the 2 Unit course. Independently they complete and are examined on the Extension 1 course, but taking that extra 1 Unit should not affect how they are assessed on the original 2 Unit course.
- These remarks should also apply to the school assessments.

THE GENERAL/2 UNIT HSC EXAMINATION OVERLAP:
Here the proposal is for a 30% overlap:

- Again, without control of 'time on task' by students during the examination, any scaling based on these common questions would be invalid.
- Because General students are not doing calculus, the common questions are going to come from parts of the syllabus well away from the central focus of the 2 Unit students.
- After studying calculus, students have a completely different experience of mathematics. Assessing the two groups with the one assessment instrument makes no sense.
- In particular, the appropriate language in which questions are couched is, by necessity, quite different for the two groups.

THE PROBLEM OF THE ATAR SCALING OF THE GENERAL AND 2 UNIT COURSES:
This serious problem has arisen since the new HSC in 2001 because of University decisions, not Board decisions. A series of meetings last year was organised by David Whitehouse of Shore School. These have clarified several reasons for the problem, and there seems to be slow progress towards a permanent solution. The causes and possible remedies are:

- The most important initial cause was probably the universities' dropping of pre-requisites — in particular, for economics and the sciences, 2 Unit mathematics is only needed as 'assumed knowledge'. This was misinterpreted, and able students began choosing the less demanding General Mathematics. Some progress seems to have been made, with Sydney University soon to bring back that particular pre-requisite, but it will have little effect unless adopted by the other leading universities.
- Also significant was the dropping of the 'breadth requirement', requiring that the ATAR include a unit of English, a unit of mathematics/science, and a unit of another humanity. This led some able students to drop 2 Unit Mathematics in favour of a pure humanities HSC. There have been some moves to insist on mathematics/science in the HSC, although this would still not be a return to the earlier 'liberal education' concept of the ATAR.
- From the early 2000s, there has been a tendency for the 2 Unit examination to be too difficult for the 2 Unit students. That was motivated by the desire to separate the 3 Unit students on the 2 Unit paper, but it has discouraged many students from taking 2 Unit. There have been many complaints, and that problem now seems to have been fixed.
- The ATAR Technical Scaling Committee has recently looked at their piggy-back algorithms and come up with a very promising tweak that could end the present bias towards General.

- There is an unfortunate instability. Once movement of able students from 2 unit to General started, and the outcomes were perceived as successful, then the movement increased more the next year. Reversing this will require convincing people that the system has changed.

There was general agreement, however, that fiddling with the examinations using overlap questions, or 'relating' the general and 2 Unit papers, or using a separate paper for 3 Unit students, would not be advisable, for roughly the reasons outlined above. While the scaling problem needs urgent attention, I would therefore recommend that the Board not adopt such measures, but instead join with the mathematics teaching community and make representations to the universities to get on with the job of correcting the instabilities that they themselves have caused.

2. Technology and projects as teaching and assessment tools
TEACHING TOOLS: These things are widely seen as distractions from mathematical reasoning, and in most classes are a waste of time. By all means encourage teachers to use whatever technology they have available (if it is in working order), because all teachers teach best when they are following their own vision of their subject, but mandating technology is bad classroom practice.

Similarly, a teacher who is committed to a particular project may find that using this in a sustained way over a few days will help motivate the class, and such a teacher should be encouraged to do so, but other teachers will want to motivate their students in other ways, in particular from the logic of the subject itself.

These are matters for teachers and subject masters, very dependent on the resources of each school and the interests and motivations of its students. The Board should get out of the mathematics classroom unless the school has demonstrably poor practices.

ASSESSMENT TOOLS: Using projects as assessment tools in these courses seems to have been a dogmatic decision taken with little understanding either of mathematics or of the mathematics classroom.

- In some disciplines, such as music and art, projects are clearly part of the assessment process. But each discipline is different, with its own distinctive language and structure, and has its own effective assessment methods. There is no one assessment method to fit all disciplines. Projects may well be seen by some mathematics teachers as satisfactory classroom tools, but they have no place as mandated assessment components.
- An assessment tool should be assessing proof, understanding and problem-solving. These things are well tested in written examinations, where apart from family emergencies, a student's marks are typically stable, or rising quietly, or falling quietly, over the six years of secondary school. They are very reliable assessment instruments.
- HSC courses with projects already have huge problems with plagiarism, with assignments being traded on the web, and assignments being completed by tutors or parents. It would be far more difficult to notice and identify such practices with mathematics projects than with projects of most other disciplines. These things would certainly occur routinely, leading to rank unfairness, accusations true and false within classes and cohorts, and great bitterness by students who are not cheating. Have the writers not experienced the crushing emotional pressures of the HSC students in Year 12?
- Projects would cheapen mathematics in the eyes of most students in these courses. At the moment, they have a deep respect for the transparency, fairness and thoroughness of the mathematics examinations, and this contributes to their high regard for mathematics. Projects would lead to boredom and contempt.

3. Examining the Year 11 material

It was a bad decision by the Board some years ago to restrict the HSC examination of Year 11 material in any way. The three courses are each a unity, and each should be examined as such. Obviously examiners will concentrate on Year 12 material because most of the Year 11 material is contained within it, but they always want to examine students' ability to understand the unity of the courses and apply methods from different parts to the one problem.

The Drafts go much further than the current guidelines, making the bald statement that 'Preliminary/Year 11 material will not be assessed in the HSC examination'.

- Taken literally, the sentence is absurd, because it means that every time a student replaces $\sin 30°$ by $\frac{1}{2}$ within a long question, any mistake in that detail would not be penalised. Did the writers intend this? Would barristers argue this in court?
- Does it mean that a student can forget about arithmetic sequences and the definition of the derivative as a limit because they are not mentioned in a Year 12 course? I fear that this may actually be intended, in which case the unfortunate result would be further fragmentation of the courses.
- GPs are mentioned again in the 2 Unit Year 12 'Financial Mathematics' section (page 61). Does this mean that GPs can only be examined in this extremely difficult context?

These Drafts seem determined to dismember the unity of our subject in every way possible.

4. Short answer questions

It is widely believed that the purpose the Board's unfortunate decision to include multiple choice questions in the HSC examination was saving money. These questions have upset very badly the flow of the examinations papers. They are widely disliked, and should be removed.

- Before these questions were introduced, all three papers, but the 2 Unit paper particularly, began with very straightforward questions in coordinate geometry, trigonometry and basic calculus. Students knew that they could do well here, and going steadily through them calmed nerves and built confidence. Now they have to cope with the jumpiness of often quite difficult multiple choice questions.
- A multiple choice question does not control how the candidate is working. To take a very simple example, examiners may want to test division by the question $203 \div 7$, with possible answers 18, 21, 26, 29, 40:
  — DIVISION: One student will perform the division.
  — MULTIPLICATION: Another will work backwards, multiplying each answer by 7.
  — INEQUALITIES: Another will eliminate all but 26 and 29 by size.
  — MODULO 10 ARITHMETIC: Another will realise that the answer has to end in 9.
  — PRIME FACTORISATION: Another will rule out even numbers and multiples of 3.
  There is no control on which of five distinct processes of arithmetic is being examined.
- Setting out a logical argument in good mathematical prose is the ideal that we are teaching for students' answers, whether in examinations or exercises. Multiple choice questions encourage the scribbling of jumbled fragments on scraps of paper, with the rushed student knowing that his working will never be seen. Thus the Board's examination is openly sabotaging what a student's teachers have been insisting on about mathematical language over the last six years.
- These 'pick-a-box' question encourage guessing, or perhaps guessing after a couple of eliminations, which is the opposite of what we are teaching in mathematics. They also give an impression of unfairness because 'he guessed correctly and I guessed wrongly'.

Part 5: Conclusion

Here are ten demands that one should make of a new senior mathematics course intended to prepare students for university:

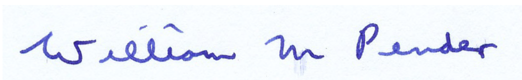
- It is unified, with everything related to everything else, as in mathematics itself.
- It strives to teach proof, understanding and problem-solving in every topic.
- Its mathematics is correct, coherent and imaginative.
- It inspires a love of and enthusiasm for mathematics.
- It shows an understanding of the range of abilities of the students taking it.
- It teaches a substantial body of mathematics suitably challenging for its students.
- It prepares students appropriately for the mathematics demands of university courses.
- Its own language is excellent mathematical prose — precise and straightforward.
- It takes care to teach students the correct use of mathematical language.
- It is written in the full awareness of, and respect for, the previous syllabuses, and for the achievements of the many teachers, students and academics associated with them.

Each of the three present courses meets all these ten demands very well, and has been very well accepted by the mathematics teaching community and the students. Each Draft course, however, meets none of these ten demands. It is a stark contrast, and as I have said above, the Board should not be taking them seriously as a basis on which to prepare final versions of new syllabuses because they are too poor to be repaired and have nothing to offer.

The present courses are wonderful, and there is no reason to change them. But if the Board wants to review these courses, the starting point is to take the present courses as drafts. If genuine mathematicians appointed by the professors of the leading universities can improve them according to the criteria above, then everyone will gladly accept such changes, as they have in the past.

We all rely very heavily on the Board to protect us from whatever forces — ACARA or other States perhaps — were responsible for these disastrous Drafts.

Yours sincerely,



Dr Bill Pender
Retired Subject Master in Mathematics, Sydney Grammar School

Cc:
Anna Wethereld
Board Inspector, Mathematics and Numeracy
anna.wethereld@bostes.nsw.edu.au