Sydney Boys High School

4 unit mathematics

Trial HSC Examination 1994

1. (a) Find $\int \frac{2x}{1+x^4}\ dx$

(b) Evaluate

(i) $\int_1^2 \frac{dx}{x^2-x+1}$

(ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$

(iii) $\int_0^1 e^{\sqrt{x}}\ dx$

(c) Prove that $\int_a^b f(mx)\ dx = \frac{1}{m}\int_{ma}^{mb} f(x)\ dx \quad (m \neq 0)$

2. (a) (i) If $z = 3 - 4i$, express z^2 and $\frac{1}{z}$ in the form $a + ib$ where a and b are real numbers. Represent them on an Argand diagram.

(ii) If $w^2 = z$, express the two values of w in the form $a + ib$.

(b) Illustrate on the Argand diagram the region $\{z : 0 \leq \arg(z+4) \leq \frac{2\pi}{3} \wedge |z+4| \leq 4\}$

(c) Let z_1, z_2 and z_3 be three complex numbers represented by Z_1, Z_2 and Z_3 respectively where $z_1 \times z_3 = (z_2)^2$. Show that OZ_2 bisects $\angle Z_1OZ_3$.

(d) A sequence $z_1, z_2, z_3, \ldots$ satisfy $z_{m+1} = z_n^2 + z_1$ for all $n \geq 1$. If $z_1 = i$, find the distinct values that occur in the sequence.

3. (a) A hyperbola has equation $9x^2 - 16y^2 = 144$

(i) Prove that the eccentricity is $\frac{5}{4}$.

(ii) Find the coordinates of the foci.

(iii) Find the equations of the asymptotes.

(iv) Find the length of the latus rectum. (A latus rectum is a focal chord parallel to the directrix.)

(v) Find the equation of the normal to the hyperbola at the point $(\frac{4\sqrt{10}}{3}, 1)$

(b) The tangents at the two points P and Q with parameters θ and ϕ respectively on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect at the point T. Show that
$T = \left(\frac{a(\sin\theta - \sin\phi)}{\sin(\theta-\phi)}, \frac{b(\cos\phi - \cos\theta)}{\sin(\theta-\phi)}\right)$.

(c) Find the condition for the line $px + qy + r = 0$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

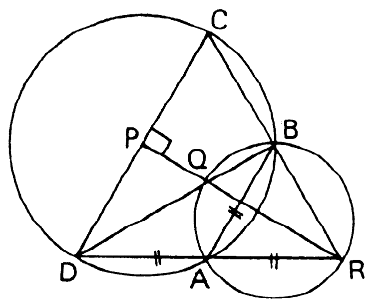
4. (a) Sketch, on separate diagrams, the curves

(i) $y = \frac{x-4}{x}$ **(ii)** $y = \frac{x^2-16}{x^2}$.

(The equation of any asymptotes should be stated, together with the coordinates of any intersections with the axes.) Hence or otherwise sketch the curves

(iii) $y = |\frac{x-4}{x}|$ **(iv)** $y^2 = \frac{x^2-16}{x^2}$

(b) $AB = AD = AR,\ RP \perp DC$.



Prove that
(i) $BCPQ$ is a cyclic quadrilateral.
(ii) $\angle CBD = 90°$
(iii) $AB = AP$.

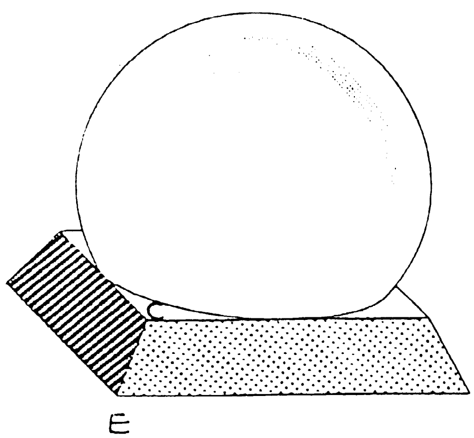
5. (a) Let $f(x) = (\ln x)^2 + 5$
(i) Find the domain of f.
(ii) Find $f'(x)$
(iii) Sketch the graph
(iv) Find the maximum domain such that $f(x)$ has an inverse function
(v) Find the inverse function f^{-1} of f and state its domain and range
(vi) Sketch the graph of f^{-1}
(b) The sequence of real numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5$ and $u_{n+1} = (u_n + \frac{1}{i_n})^2$ for all $n \geq 1$. Prove by induction that, for every positive integer n, $u_n > 2^m$ where $m = 2^n$.

6. (a) For the equation $x^4 + 2x^3 + 3x^2 + 5x + 1 = 0$
(i) Obtain the sum of the squares of the roots of the equation
(ii) Show that the equation has two negative roots, α and β, such that $-2 < \alpha < \beta < 0$
(iii) Hence, or otherwise, prove that the equation has no other real roots.
(b) The roots of the equation $x^3 + px + q = 0$, $(q \neq 0)$ are α, β and γ
(i) Show that $\alpha^{n+3} + p\alpha^{n+1} + q\alpha^n = 0$ where n is a positive integer.
(ii) Write down equations involving β and γ similar to **(i)**.
(iii) Deduce that $S_{n+3} + pS_{n+1} + qS_n = 0$, where $S_n = \alpha^n + \beta^n + \gamma^n$
(iv) Hence show that $S_3 = -3q$, and find S_5 in terms of p and q.
7. (a) Use DeMoivres' theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. Hence:
(i) Solve the equation $16x^4 - 16x^2 + 1 = 0$ and deduce the exact values of $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$.

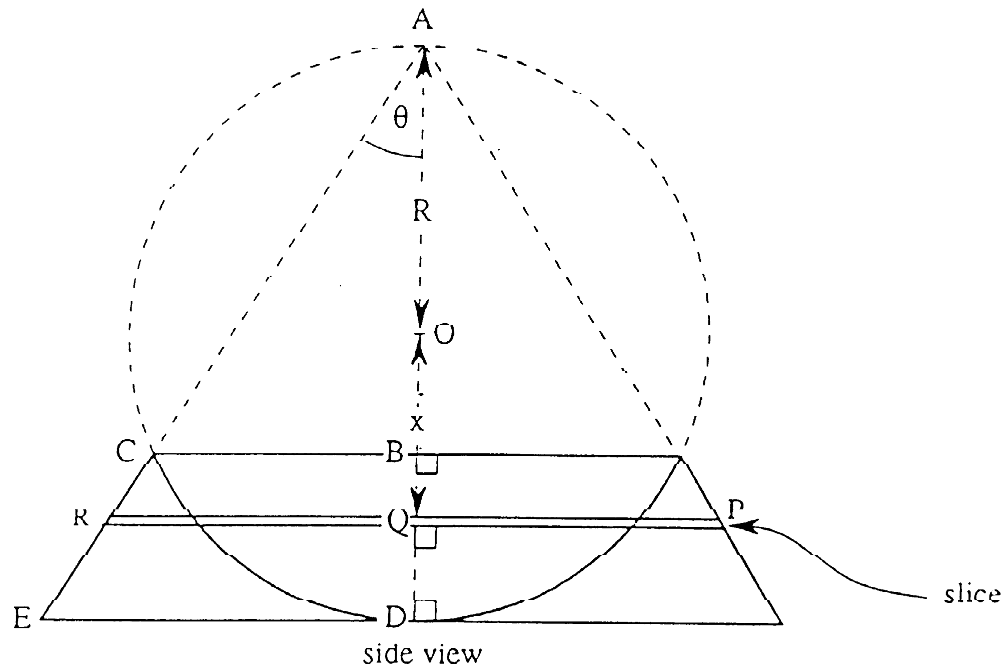
(b) **(i)** A particle P is projected, from a point O on the horizontal ground, with speed V at an angle θ above the horizontal, where $\tan\theta = \frac{1}{3}$. The particle passes through the point with coordinates $(3a, \frac{3a}{4})$ relative to the horizontal and vertical axes at O in the plane of motion. Show that $v^2 = 20ga$.

(ii) A particle Q is projected, from a point O at the instant when P is moving horizontally. It strikes the ground at the same place and at the same instant as P. Show that the speed of projected of Q is $\sqrt{\frac{145ga}{2}}$ and find the tangent of the angle of projection.

8. **(a)**



Mr Keating's crystal ball rests on a solid stand which is in the shape of a square based frustrum as shown.



The stand is constructed such that the crystal ball of radius R fits snugly inside and just touches the centre of the square base. The sides, EC, of the base slope so that, if extended, they would pass through the top-most point on the ball at A and make an angle θ with the vertical AD.

(i) Show that $OB = R\cos 2\theta$

(ii) Consider a slice PQR of thickness Δx as shown taken perpendicular to AD such that $OQ = x$ units. Draw a neat sketch of the slice, determine its dimensions and show that it has a volume ΔV given by $\Delta V \approx [4\tan^2\theta(R+x)^2 - \pi(R^2 - x^2)]\Delta x$

(iii) Find the volume of such a solid where the angle $\theta = \frac{\pi}{6}$.