

QUESTION ONE (15 marks) Use a separate writing booklet.

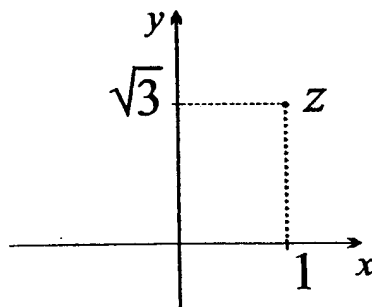
Marks

- (a) Find $\int_0^4 \frac{1}{\sqrt{2x+1}} dx$. 2
- (b) Find $\int \tan^3 x \sec^2 x dx$. 2
- (c) Find $\int \frac{x}{x^2 - 4x + 8} dx$. 3
- (d) (i) Find the values of A and B such that $\frac{3x^2 - 10}{x^2 - 4x + 4} = 3 + \frac{A}{x - 2} + \frac{B}{(x - 2)^2}$. 2
- (ii) Find $\int \frac{3x^2 - 10}{x^2 - 4x + 4} dx$. 2
- (e) Use integration by parts twice, to show that $\int_1^e \sin(\ln x) dx = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$. 4

QUESTION TWO (15 marks) Use a separate writing booklet.

Marks

- (a) Simplify $|\cos \theta + i \sin \theta|$. 1
- (b) Express $\frac{i^5(1-i)}{2+i}$ in the form $a + ib$ where a and b are rational. 2
- (c) By drawing a diagram, or otherwise, find the solutions of $z^5 = -1$. 2
- (d) Graph the region in the Argand diagram which simultaneously satisfies 3
- $1 \leq |z - i| \leq 2$ and $\text{Im } z \geq 0$.
- (e) Find the complex number ϕ if $1 + i$ is a root of the equation $z^2 + \phi z - i = 0$. 2
- (f)

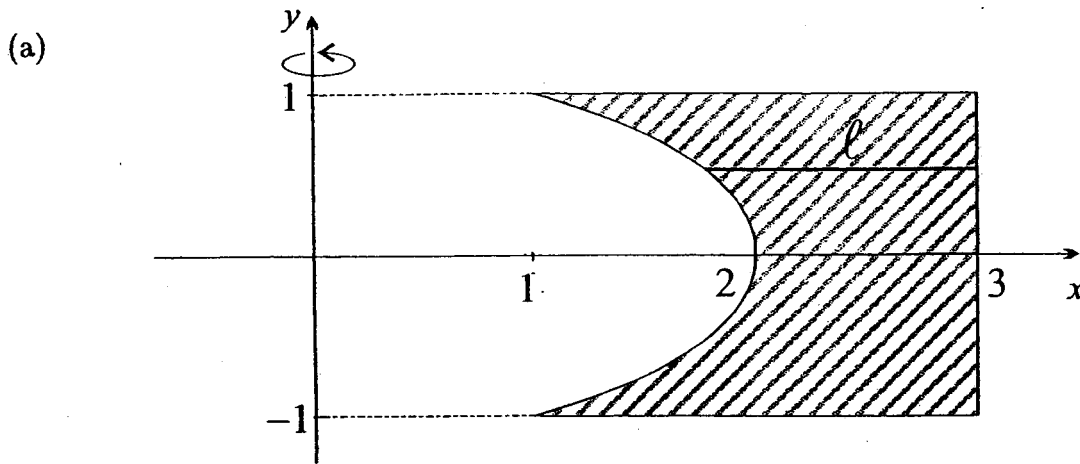


Suppose that $z = 1 + \sqrt{3}i$ and $\omega = (\text{cis } \alpha)z$ where $-\pi < \alpha \leq \pi$.

- (i) Find the argument of z . 1
- (ii) Find the value of α if ω is purely imaginary and $\text{Im}(\omega) > 0$. 2
- (iii) Find the value of $\arg(z + \omega)$ if ω is purely imaginary and $\text{Im}(\omega) > 0$. 2

QUESTION THREE (15 marks) Use a separate writing booklet.

Marks



The diagram above shows the region bounded by the curve $x = 2 - y^2$ and the lines $x = 3$, $y = 1$ and $y = -1$. This region is rotated about the y -axis to form a solid. The interval ℓ at height y sweeps out an annulus.

- (i) Show that the annulus at height y has area equal to

2

$$\pi(5 + 4y^2 - y^4).$$

- (ii) Find the volume of the solid.

2

- (b) Consider the function $f(x) = \frac{1}{1+x^3}$.

- (i) Show that there is a horizontal point of inflexion at $x = 0$.

2

- (ii) Find the vertical asymptote and the horizontal asymptote.

2

- (iii) Sketch $y = f(x)$ showing the features from parts (a) and (b) and the y -intercept.

2

- (iv) On a separate diagram sketch $y = |f(x)|$.

1

- (v) On a separate diagram sketch $y^2 = f(x)$.

2

- (vi) On a separate diagram sketch $y = e^{f(x)}$.

2

QUESTION FOUR (15 marks) Use a separate writing booklet.

Marks

(a) Consider the polynomial equation $x^3 - 3x^2 + x - 5 = 0$ which has roots α , β and γ .

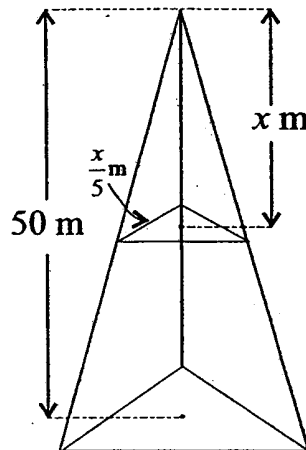
(i) Show that $\alpha + \beta = 3 - \gamma$.

1

(ii) Write down similar expressions for $\alpha + \gamma$ and $\beta + \gamma$ and hence find a polynomial equation which has the roots $\alpha + \beta$, $\alpha + \gamma$ and $\beta + \gamma$.

2

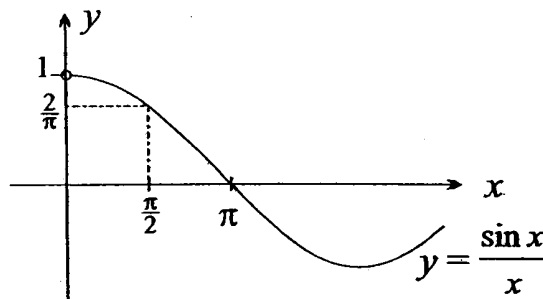
(b)



The diagram above shows a monument 50 metres high. A horizontal cross section x metres from the top is an equilateral triangle with sides $\frac{x}{5}$ metres. Use integration to find the volume of the monument.

3

(c)



Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by the curve $y = \frac{\sin x}{x}$ and the lines $y = 0$ and $x = \frac{\pi}{2}$ is rotated about the y -axis.

4

(d) An hyperbola is defined parametrically by $x = 3 \sec \theta$ and $y = 4 \tan \theta$.

(i) Write the equation of the curve in Cartesian form and show that the eccentricity is $\frac{5}{3}$.

2

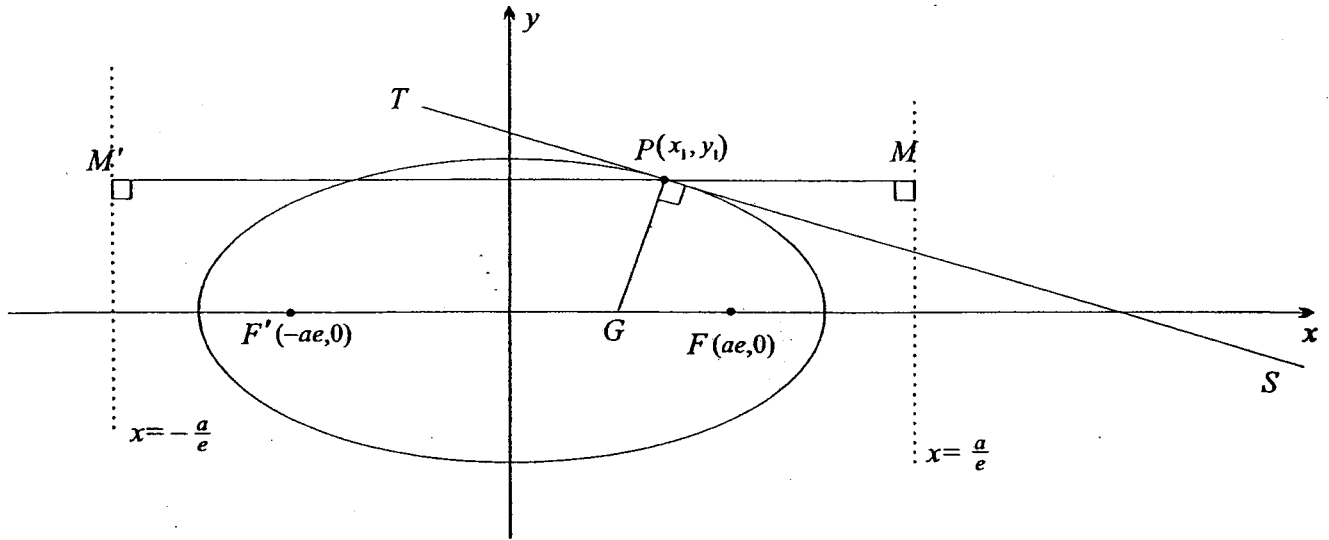
(ii) Sketch the curve showing its x -intercepts, foci, directrices and asymptotes.

4

QUESTION FIVE (15 marks) Use a separate writing booklet.

Marks

(a)



The diagram above shows the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $F(ae, 0)$ and $F'(-ae, 0)$.

$P(x_1, y_1)$ is any point on the ellipse.

Let M and M' be the feet of the perpendiculars from P to the directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.

Line TS is a tangent to the ellipse at P and G is the point where the normal at P meets the x -axis.

(i) Show that the equation of the normal at P is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$. 3

(ii) Show that the point G has co-ordinates $(e^2x_1, 0)$. 3

(iii) Show that the distance PF is $a - ex_1$. 2

(iv) Show that $\frac{PF}{FG} = \frac{PF'}{F'G}$. 2

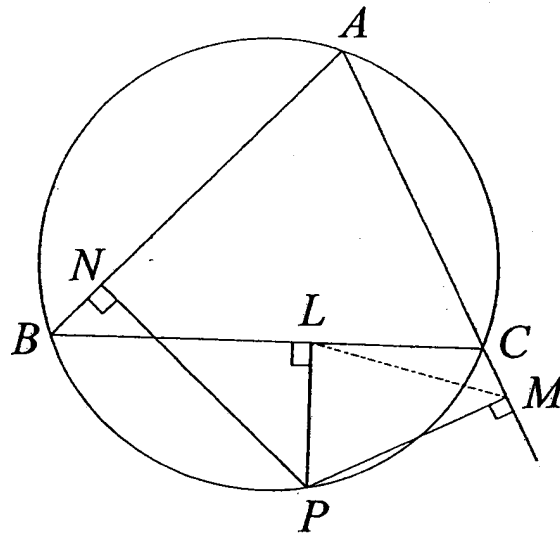
(b) (i) Show that $1 - \cos 2\theta - i \sin 2\theta = 2 \sin \theta (\sin \theta - i \cos \theta)$. 2

(ii) Given that $\frac{z-1}{z} = \text{cis } \frac{2\pi}{5}$, show that $z = \frac{1}{2}(1 + i \cot \frac{\pi}{5})$. 3

QUESTION SIX (15 marks) Use a separate writing booklet.

Marks

(a)



In the diagram above, ABC is a triangle with the circumcircle through points A, B and C drawn. P is another point on the minor arc BC . Points L, M and N are the feet of the perpendiculars from P to the sides BC, CA and AB respectively.

(i) Copy the diagram and explain why P, L, N and B are concyclic. 1

(ii) Explain why P, L, C and M are concyclic. 1

(iii) Let $\angle PLM = \alpha$.

(α) Show that $\angle ABP = \alpha$. 2

(β) Hence show that M, L and N are collinear. 2

(b) A particle of unit mass is thrown vertically downwards with an initial velocity of v_0 . It experiences a resistive force of magnitude kv^2 where v is its velocity. Taking downwards as the positive direction, the equation of motion of the particle is given by

$$\ddot{x} = g - kv^2.$$

Let V be the terminal velocity of the particle.

(i) Explain why $V = \sqrt{\frac{g}{k}}$. 1

(ii) Show that $v^2 = V^2 + (v_0^2 - V^2)e^{-2kx}$. 4

(c) Let $z = x + iy$ be any non-zero complex number such that $z + \frac{1}{z} = k$, where k is a real number.

(i) Prove that either $y = 0$ or $x^2 + y^2 = 1$. 2

(ii) Show that if $y = 0$ then $|k| \geq 2$. 2

Exam continues next page ...

QUESTION SEVEN (15 marks) Use a separate writing booklet.

Marks

- (a) (i) Write down $\cos 2\theta$ in terms of $\tan \theta$. 1
- (ii) Show that $\cos 4\theta = \frac{1 - 6 \tan^2 \theta + \tan^4 \theta}{1 + 2 \tan^2 \theta + \tan^4 \theta}$. 3
- (iii) Deduce that $\tan^2 \frac{\pi}{8} + \tan^2 \frac{3\pi}{8} = 6$. 3
- (b) Consider the equation $z^7 = 1$.

This equation has seven roots $1, \rho, \rho^2, \dots, \rho^6$, where $\rho = \text{cis } \frac{2\pi}{7}$.

Let $\alpha = \rho + \rho^2 + \rho^4$ and $\theta = \rho^3 + \rho^5 + \rho^6$.

- (i) Express ρ^9 as a lower positive power of ρ . 1
- (ii) Simplify $\alpha + \theta$. 2
- (iii) Simplify $\alpha\theta$. 2
- (iv) Form a quadratic equation with α and θ as roots. 1
- (v) Deduce that $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{8\pi}{7} = -\frac{1}{2}$. 2

QUESTION EIGHT (15 marks) Use a separate writing booklet.

Marks

- (a) The sequence a_1, a_2, \dots, a_n is defined by $a_n = \frac{(2n)!}{2^n n!}$. 4

Show by induction on n that a_n is an odd positive integer.

- (b) Suppose that $y = f(x)$ is an increasing function for $x \geq 1$.
Suppose also that $f(x) \geq 0$ for $x \geq 1$.

(i) Explain, with the aid of a diagram, why 2

$$f(1) + f(2) + \dots + f(n-1) < \int_1^n f(x) dx < f(2) + f(3) + \dots + f(n).$$

(ii) Show that $\int_1^n \ln x dx = n \ln n - n + 1$. 2

(iii) Use parts (i) and (ii) to deduce that, for $n > 1$:

(α) $n! > \frac{n^n}{e^{n-1}}$ 3

(β) $n! < \frac{n^{n+1}}{e^{n-1}}$ 2

(iv) Find $\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n}$. (You may assume that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.) 2

END OF EXAMINATION