DRAFT SENIOR SECONDARY CURRICULUM – SPECIALIST MATHEMATICS

Organisation

1. Overview of senior secondary Australian Curriculum

ACARA has developed draft senior secondary Australian Curriculum for English, Mathematics, Science and History according to a set of design specifications (see http://www.acara.edu.au/curriculum/development_of_the_australian_curriculum.html). The ACARA Board approved these specifications following consultation with state and territory curriculum, assessment and certification authorities.

Senior secondary Australian Curriculum will specify content and achievement standards for each senior secondary subject. Content refers to the knowledge, understanding and skills to be taught and learned within a given subject. Achievement standards refer to descriptions of the quality of learning (the depth of understanding, extent of knowledge and sophistication of skill) demonstrated by students who have studied the content for the subject.

The senior secondary Australian Curriculum for each subject has been organised into four units. The last two units are cognitively more challenging than the first two units. Each unit is designed to be taught in about half a 'school year' of senior secondary studies (approximately 50–60 hours duration including assessment). However, the senior secondary units have also been designed so that they may be studied singly, in pairs (that is, year-long), or as four units over two years. State and territory curriculum, assessment and certification authorities are responsible for the structure and organisation of their senior secondary courses and will determine how they will integrate the Australian Curriculum content and achievement standards into courses. They will also provide any advice on entry and exit points, in line with their curriculum, assessment and certification requirements.

States and territories, through their respective curriculum, assessment and certification authorities, will continue to be responsible for implementation of the senior secondary curriculum, including assessment, certification and the attendant quality assurance mechanisms. Each of these authorities acts in accordance with its respective legislation and the policy framework of its state government and Board. They will determine the assessment and certification specifications for their courses that use the Australian Curriculum content and achievement standards and any additional information, guidelines and rules to satisfy local requirements.

These draft documents should not, therefore, be read as proposed courses of study. Rather, they are presented as draft content and achievement standards that will provide the basis for senior secondary curriculum in each state and territory in the future. Once approved, the content and achievement standards would subsequently be integrated by states and territories into their courses.

2. Senior Secondary Mathematics subjects

The Senior Secondary Australian Curriculum: Mathematics consists of four subjects in mathematics. The subjects are differentiated, each focusing on a pathway that will meet the learning needs of a particular group of senior secondary students. Each subject is organised into four units.

Essential Mathematics focuses on using mathematics effectively, efficiently and critically to make informed decisions. It provides students with the mathematical knowledge, skills and understanding to solve problems in real contexts for a range of workplace, personal, further learning and community settings. This subject provides the opportunity for students to prepare for post-school options of employment and further training.

General Mathematics focuses on using the techniques of discrete mathematics to solve problems in contexts that include financial modelling, network analysis, route and project planning, decision making, and discrete growth and decay. It provides an opportunity to analyse and solve a wide range of geometrical problems in areas such as measurement, scaling, triangulation and navigation. It also provides opportunities to develop systematic strategies based on the statistical investigation process for answering statistical questions that involve comparing groups, investigating associations and analysing time series.

Mathematical Methods focuses on the development of the use of calculus and statistical analysis. The study of calculus in Mathematical Methods provides a basis for an understanding of the physical world involving rates of change, and includes the use of functions, their derivatives and integrals, in modelling physical processes. The study of statistics in Mathematical Methods develops the ability to describe and analyse phenomena involving uncertainty and variation.

Specialist Mathematics provides opportunities, beyond those presented in Mathematical Methods, to develop rigorous mathematical arguments and proofs, and to use mathematical models more extensively. Specialist Mathematics contains topics in functions and calculus that build on and deepen the ideas presented in Mathematical Methods as well as demonstrate their application in many areas. Specialist Mathematics also extends understanding and knowledge of probability and statistics and introduces the topics of vectors, complex numbers, matrices and recursive methods. Specialist Mathematics is the only mathematics subject that cannot be taken as a stand-alone subject.

3. Structure of Specialist Mathematics

Specialist Mathematics is structured over four units. The topics in Unit 1 broaden students' mathematical experience and provide different scenarios for incorporating mathematical arguments and problem solving. The unit provides a blending of algebraic and geometric thinking. In this subject there is a progression of content, applications, level of sophistication and abstraction. For example, vectors for 2-dimensional space are introduced in Unit 1 and then in Unit 3 they are studied for 3-dimensional space. The Unit 3 Vector topic leads to the establishment of the equations of lines and planes, and this in turn prepares students for the solution of simultaneous equations in three variables that is part of the Matrices topic. As a further example of this progression, the study of calculus, which is developed in Mathematical Methods and Unit 3 of Specialist Mathematics, is applied in Vectors in Unit 3 and Applications of Calculus in Unit 4.

	Unit 1	Unit 2	Unit 3	Unit 4
Specialist Mathematics	Recurrence Relations Combinatorics Geometry Vectors in the plane	Trigonometry Matrices Real and Complex Numbers Graph Theory	Vectors in three dimensions Matrices and Systems of Equations Complex numbers Functions and Calculus	Further Calculus and Applications of Calculus Statistical Inference for Continuous Data

Units

Unit 1 contains four topics that complement the content of Mathematical Methods. The proficiency strand, 'Reasoning', of the F-10 curriculum is continued explicitly in the topic 'Geometry' through a discussion of developing mathematical arguments. This topic also provides the opportunity to summarise and extend students' studies in Euclidean Geometry. This understanding is of great benefit in the later study of topics such as vectors and complex numbers. Two of the topics in this unit introduce students to some discrete mathematics topics. The topic 'Recursion' is a key idea in many areas and is employed widely in the community, for example in the calculation of loan repayments and in spreadsheets. Recursion is also applied in the construction of proofs by the principle of mathematical induction. The topic 'Combinatorics' provides techniques that are very useful in many areas of mathematics, including probability, graph theory and algebra. The topic 'Vectors' provides new perspectives of working with 2-dimensional space and serves as an introduction to techniques which can be extended to 3-dimensional space in Unit 3. These four topics considerably broaden students' mathematical experience and therefore begin an awakening to the breadth and utility of the subject. They also enable students to increase their mathematical flexibility and versatility.

Unit 2 contains two topics, 'Matrices' and 'Real and complex numbers', that are extended in Unit 3. The topic 'Trigonometry' contains techniques that are used in other topics in both this unit and Unit 3. All of these topics develop students' ability to construct mathematical arguments. 'Matrices' provides new perspectives for working with 2-dimensional space and this topic serves as an introduction to techniques which will be extended to 3-dimensional space in Unit 3. The 'Real and complex numbers' topic provides a continuation of the study of numbers. 'Graph Theory' is an important area of study in discrete mathematics and has many applications. It gives students further opportunities to construct mathematical proofs in different contexts. The arguments can involve algebraic, combinatoric and geometric techniques.

Unit 3 contains four topics, 'Vectors', 'Matrices and systems of equations', 'Complex numbers' and 'Functions and calculus'. The study of matrices and vectors began in Units 1 and 2, In Unit 1, vectors in 1 and 2-dimensional space were the focus. In this unit, 3-dimensional vectors are studied, and vector equations and vector calculus are introduced, with the latter extending students' knowledge of calculus from Mathematical Methods. Cartesian and vector equations of lines, together with equations of planes, enable students to work with geometric considerations as well as motion in 3-dimensional space. In Unit 2, the emphasis for matrices is on the solution of simultaneous equations in two variables and transformations of the plane. In this unit, the emphasis is on the solution of systems of equations with three variables and the geometric interpretation of these solutions. The Cartesian form of complex numbers was introduced in Unit 2 and the study of complex numbers is now extended to the polar form. Finally, the study of functions and techniques of calculus begun in Mathematical Methods are extended and utilised in the sketching of graphs and the solution of problems involving integration.

Unit 4 contains two topics, 'Further calculus and applications of calculus' and 'Statistical inference for bivariate continuous data'. In Unit 4, the study of differentiation and integration of functions is continued and the techniques developed from this and previous topics in calculus are applied in the area of simple differential equations, in particular in biology and kinematics. These topics serve to demonstrate the applicability of the mathematics learnt throughout this course. In this unit, all of the students' previous experience in statistics is drawn together in the study of statistical inference for bivariate continuous data. This is a topic that demonstrates the utility and power of statistics.

Organisation of achievement standards

The achievement standards have been organised into two dimensions, 'Concepts and Techniques' and 'Reasoning and Communication'. These two dimensions reflect students' understanding and skills in the study of mathematics.

Role of technology

It is assumed that students will be taught the Senior Secondary Australian Curriculum: Mathematics subjects with an extensive range of technological applications and techniques. If appropriately used, these have the potential to enhance the teaching and learning of mathematics. However, students also need to continue to develop skills that do not depend on technology. The ability to be able to choose when or when not to use some form of technology and to be able to work flexibly with technology are important skills in these subjects.

4. Links to F-10

For all content areas of Specialist Mathematics, the proficiency strands of the F-10 curriculum are still very much applicable and should be inherent in students' learning of the subject. The strands of understanding, fluency, problem solving and reasoning are essential and mutually reinforcing. For all content areas, practice allows students to achieve fluency of skills, such as finding the scalar product of two vectors, or finding the area of a region contained between curves, and frees up working memory for more complex aspects of problem solving. In Specialist Mathematics, the formal explanation of reasoning through mathematical proof takes an important role and the ability to present the solution of any problem in a logical and clear manner is of paramount importance. The ability to transfer skills learned to solve one class of problems, for example integration, to solve another class of problems, such as those in biology, kinematics or statistics, is a vital part of mathematics learning in this subject. In studying Specialist Mathematics, it is desirable that students complete topics from 10A. The knowledge and skills from the content descriptions ACMMG273, ACMMG274, ACMSP279 from 10A are highly recommended as preparation for Specialist Mathematics.

5. Representation of General Capabilities

The seven general capabilities of *Literacy*, *Numeracy*, *Information and communication technology (ICT) capability*, *Critical and creative thinking*, *Personal and social capability, Ethical behaviour,* and *Intercultural understanding* are identified where they offer opportunities to add depth and richness to student learning. Teachers will find opportunities to incorporate explicit teaching of the capabilities depending on their choice of learning activities.

General capabilities that are specifically covered in *Specialist Mathematics* include *Literacy, Numeracy, Information and communication technology (ICT) capability, Critical and creative thinking and Ethical behaviour.*

Literacy is of fundamental importance in students' development of Specialist Mathematics as they develop the knowledge, skills and dispositions to interpret and use language confidently for learning. Students will be taught to read, understand and gather information presented in a wide range of genres, modes and representations (including text, symbols, graphs and tables). They are taught to communicate ideas logically and fluently and to structure arguments and proofs.

Numeracy involves students recognising and understanding the role of mathematics in the world and to use mathematical knowledge and skills purposefully. Specialist Mathematics provides the opportunity to apply mathematical understanding and skills in real world contexts. The twenty-first century world is information driven and through reasoning and analysis, students can make informed judgements.

Critical and creative thinking is inherent in Specialist Mathematics. Students develop their critical and creative thinking as they learn to generate and evaluate knowledge, clarify concepts and ideas, seek possibilities, consider alternatives and solve problems. Critical and creative thinking is integral to activities that require students to think broadly and deeply using skills, behaviours and dispositions such as reason, logic, resourcefulness, imagination and innovation in all learning areas at school and their lives beyond school.

Ethical behaviour involves students exploring the ethics of their own and other others' actions. Students develop the capability to behave ethically as they identify and investigate the nature of ethical concepts, values, character traits and principles, and understand how reasoning can assist ethical judgement. There are opportunities in Specialist Mathematics to explore, develop and apply ethical behaviour in a range of contexts.

Information and communication technology (ICT) is a key part of Specialist Mathematics. Students develop ICT capability as they learn to use ICT effectively and appropriately to access, create and communicate information and ideas, solve problems, perform calculations, draw graphs, collect, analyse and interpret data. Digital technologies can engage students and promote the understanding of key concepts.

There are also opportunities within Specialist Mathematics to develop the general capabilities of *Intercultural understanding* and *Personal and social capability*, with an appropriate choice of activities and contexts provided by the teacher.

6. Representation of Cross-curriculum priorities

The Cross-Curriculum Priorities of Aboriginal and Torres Strait Islander histories and cultures, Asia and Australia's engagement with Asia, and Sustainability, are not overtly evident in the content descriptions of the Specialist Mathematics subject. However opportunities exist for teachers to reference them in the context of their teaching of relevant topics, especially those topics which use real data to develop mathematical and statistical concepts.

Aboriginal and Torres Strait Islander histories and cultures

Students will deepen their understanding of the lives of Aboriginal and Torres Strait Islander Peoples through the application of mathematical concepts in appropriate contexts. Teachers could develop statistical and mathematical learning opportunities based on information and data pertinent to Aboriginal and Torres Strait Islander histories and cultures.

Asia and Australia's engagement with Asia

In Specialist Mathematics, the priority of Asia and Australia's engagement with Asia provides rich and engaging contexts for developing students' mathematical knowledge, skills and understanding. In Specialist Mathematics , students develop mathematical understanding by drawing on knowledge of and examples from the Asia region. Investigations and analysis can be used to examine issues pertinent to the Asia region.

Sustainability

In Specialist Mathematics, the priority of sustainability provides rich, engaging and authentic contexts for student learning. Specialist Mathematics provides opportunities for students to develop problem solving and reasoning essential for the exploration of sustainability issues and their solutions. Mathematical understandings and skills are necessary to measure, monitor and quantify change in social, economic and ecological systems over time.

DRAFT SENIOR SECONDARY CURRICULUM – SPECIALIST MATHEMATICS

Rationale

Mathematics is the study of order, relation and pattern. From its origins in counting and measuring it has evolved in highly sophisticated and elegant ways to become the language now used to describe much of the modern world. Statistics is concerned with collecting, analysing, modelling and interpreting data in order to investigate and understand real world phenomena and solve problems in context. Together, mathematics and statistics provide a framework for thinking and a means of communication that is powerful, logical, concise and precise.

Because both mathematics and statistics are widely applicable as models of the world around us, there is ample opportunity for problem solving throughout this subject. There is also a sound logical basis to Specialist Mathematics, and in mastering the subject students will develop logical reasoning skills to a high level.

Specialist Mathematics provides opportunities, beyond those presented in Mathematical Methods, to develop rigorous mathematical arguments and proofs, and to use mathematical models more extensively. Topics are developed systematically and lay the foundations for future studies in quantitative subjects in a coherent and structured fashion. Students of Specialist Mathematics will be able to appreciate the true nature of mathematics, its beauty and its functionality.

Specialist Mathematics has been designed to be taken in conjunction with Mathematical Methods. The subject contains topics in functions and calculus that build on and deepen the ideas presented in Mathematical Methods and demonstrate their application in many areas. Students will also extend their understanding and knowledge of probability and statistics. Vectors, complex numbers, matrices and recursive methods are introduced. Specialist Mathematics is designed for students with a strong interest in mathematics, including those intending to study mathematics, statistics, all sciences and associated fields, economics or engineering at university.

For all content areas of Specialist Mathematics, the proficiency strands of the F-10 curriculum are still very much applicable and should be inherent in students' learning of the subject. These are understanding, fluency, problem solving and reasoning. These are essential and mutually reinforcing. For all content areas, practice allows students to achieve fluency of skills, such as finding the scalar product of two vectors, or finding the area of a region contained between curves, and frees up working memory for more complex aspects of problem solving. In Specialist Mathematics, the formal explanation of reasoning through mathematical proof takes an important role and the ability to present the solution of any problem in a logical and clear manner is of paramount importance. The ability to transfer skills learned to solve one class of problems, for example integration, to solve another class of problems, such as those in biology, kinematics or statistics, is a vital part of mathematics learning in this subject.

Specialist Mathematics is structured over four units. The topics in Unit 1 broaden students' mathematical experience and provide different scenarios for incorporating mathematical arguments and problem solving. The unit provides a blending of algebraic and geometric thinking. In this subject there is a progression of content, applications, level of sophistication and abstraction. For example, in Unit 1 vectors for 2-dimensional space are introduced and then in Unit 3 vectors are studied for 3-dimensional space. The Unit 3 Vector topic leads to establishing the equations of lines and planes and this in turn prepares students for the solution of simultaneous equations in three variables that is part of the matrices topic. The study of calculus, which is developed in Mathematical Methods and Unit 3 of Specialist Mathematics, is applied in Vectors in Unit 3 and Applications of Calculus in Unit 4.

Aims

Specialist mathematics aims to develop students':

- understanding of concepts and techniques drawn from discrete mathematics, geometry, trigonometry, complex numbers, vectors, matrices, calculus and statistics
- ability to solve applied problems using concepts and techniques drawn from discrete mathematics, geometry, trigonometry, complex numbers, vectors, matrices, calculus and statistics
- reasoning in mathematical and statistical contexts and interpretation of mathematical and statistical information including ascertaining the reasonableness of solutions to problems
- capacity to communicate in a concise and systematic manner using appropriate mathematical and statistical language
- ability to construct proofs.

Unit 1

Unit description

Unit 1 of Specialist Mathematics contains four topics that complement the content of Mathematical Methods. The proficiency strand, 'Reasoning', of the F-10 curriculum is continued explicitly in the topic 'Geometry' through a discussion of developing mathematical arguments. This topic also provides the opportunity to summarise and extend students' studies in Euclidean Geometry. An understanding of this topic is of great benefit in the study of later topics of the course including vectors and complex numbers.

Two of the topics in this unit introduce students to some discrete mathematics topics. The topic 'Recursion' is a key idea in many areas and is employed widely in the community, for example in the calculation of loan repayments and in spreadsheets. Recursion is also applied in the construction of proofs by the principle of mathematical induction. The topic 'Combinatorics' provides techniques that are very useful in many areas of mathematics including probability, graph theory and algebra.

The topic 'Vectors' gives new perspectives of working with 2-dimensional space and serves as an introduction to techniques which can be extended to 3-dimensional space introduced in Unit 3.

These four topics considerably broaden students' mathematical experience and therefore begin an awakening to the breadth and utility of the subject. These topics also enable the student to increase their mathematical flexibility and versatility.

Learning outcomes

By the end of this unit, students:

- understand the concepts and techniques in recurrence relations, combinatorics, geometry and vectors
- apply reasoning skills and solve problems in recurrence relations, combinatorics, geometry and vectors
- communicate their arguments and strategies when solving problems
- construct proofs in a variety of contexts including algebraic and geometric
- interpret mathematical information and ascertain the reasonableness of their solutions to problems.

Content descriptions

Topic 1: Recurrence relations

Recursive definitions of sequences:

- define and generate a sequence by using a recursive relation and initial term(s)
- work with particular examples including the Fibonacci sequence, the factorial sequence, arithmetic sequences and geometric sequences
- use general first order linear recurrence relations to generate sequences
- apply first order linear recurrence relations to solve problems in areas such as finance and the study of populations

'Direct' formulas for sequences:

- establish and use direct formulas for terms of arithmetic and geometric sequences
- establish and use formulas for the partial sums of arithmetic and geometric sequences
- establish and use the formula for the sum of an infinite geometric sequence where the common ratio has absolute value less than one

An introduction to proof by mathematical induction:

- understand the nature of inductive proof including 'the initial statement' and the inductive step
- prove results for sums, such as $1+4+9\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ for any positive integer *n*
- prove divisibility results, such as $3^{2n+4}-2^{2n}$ is divisible by 5 for any positive integer n

Topic 2: Combinatorics

Permutations (ordered arrangements):

- solve problems involving permutations
- use the multiplication principle
- use factorial notation
- solve problems involving permutations involving restrictions with or without repeated objects

The Inclusion-exclusion principle for the union of two sets and three sets:

- determine and use the formulas for finding the number of elements in the union of 2 and the union of 3 sets

The pigeon-hole principle:

- solve problems and prove results using the pigeon-hole principle

Combinations (unordered selections):

- solve problems involving combinations
- use the notation $\binom{n}{r}$ or ${}^{n}C_{r}$
- Pascal's triangle: derive and use associated simple identities

Topic 3: Geometry

The nature of proof:

- implication, converse, equivalence, negation, contrapositive
- proof by contradiction
- use of the symbols for implication ($\Rightarrow$), equivalence ($\Leftrightarrow$), and equality (=)
- use of the quantifiers 'for all' and 'there exists'
- the use of examples and counter-examples in this context
- While these ideas are illustrated through deductive Euclidean geometry in this topic they reoccur throughout all of the topics of Specialist Mathematics

Circle properties and their proofs including the following theorems:

- An angle in a semicircle is a right angle.
- The angle at the centre subtended by an arc of a circle is twice the angle at the circumference subtended by the same arc.
- Angles at the circumference of a circle subtended by the same arc are equal.
- The opposite angles of a cyclic quadrilateral are supplementary.
- Chords of equal length subtend equal angles at the centre and conversely chords subtending equal angles at the centre of a circle have the same length.
- The alternate segment theorem
- When two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord.
- When a secant (meeting the circle at *A* and *B*) and a tangent (meeting the circle at *T*) are drawn to a circle from an external point *M*, the square of length of the tangent equals the product of the lengths to the circle on the secant. ($AM \times BM = TM^2$).
- Suitable converses of some of the above results. Solve problems finding unknown angles and lengths and prove further results using the results listed above

Topic 4: Vectors in the plane

Representing vectors in the plane by directed line segments:

- examples of vectors including displacement and velocity
- magnitude and direction of a vector
- represent a scalar multiple of a vector
- use the triangle rule to find the sum and difference of two vectors

Algebra of vectors in the plane:

- use ordered pair notation and column vector notation to represent a vector
- unit vectors and the perpendicular unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$
- express a vector in component form using the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$
- addition and subtraction of vectors in component form
- multiplication of a vector in component form by a scalar
- scalar (dot) product
- apply the scalar product to vectors expressed in component form
- properties of parallel and perpendicular vectors and determining if two vectors are parallel or perpendicular
- projection of vectors
- solve problems involving displacement, force and velocity involving the above concepts

Geometric proofs using vectors in the plane including:

- The diagonals of a parallelogram meet at right angles if and only if it is a rhombus.
- Midpoints of the sides of a quadrilateral join to form a parallelogram.
- The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

Unit 2

Unit description

Unit 2 of Specialist Mathematics contains two topics, 'Matrices' and 'Real and complex numbers' that are extended in Unit 3. The topic 'Trigonometry' contains techniques that are used in other topics in both this unit and Unit 3. All of these topics develop students' ability to construct mathematical arguments. 'Matrices' gives new perspectives for working with 2- dimensional space and this topic serves as an introduction to techniques which will be extended to 3-dimensional space in Unit 3. The 'Real and complex numbers' topic provides a continuation of student's study of numbers. 'Graph theory' is an important area of study in discrete mathematics and has many applications. It also gives students further opportunities to construct mathematical proofs in different contexts. The arguments can involve algebraic, combinatoric and geometric techniques.

Learning outcomes

By the end of this unit, students:

- understand the concepts and techniques in trigonometry, matrices, real and complex numbers and graph theory
- apply reasoning skills and solve problems in trigonometry, matrices, real and complex numbers and graph theory
- communicate their arguments and strategies when solving problems
- construct proofs of results
- interpret mathematical information and ascertain the reasonableness of their solutions to problems.

Content descriptions

Topic 1:Trigonometry

The basic trigonometric functions:

- find all solutions of $f\big(a(x-b)\big)=c$ where f is one of sin, cos or tan
- graph functions with rules of the form $y=f(a(x-b))$ where f is one of sin, cos, or tan

Compound angles:

- prove and apply the angle sum and difference identities
- prove and apply the double angle identities for sine, cosine and tangent.

The reciprocal trigonometric functions, secant, cosecant and cotangent:

- define the reciprocal trigonometric functions; sketch their graphs and graph simple transformations of them

Trigonometric identities:

- prove and apply the Pythagorean identities
- prove and apply the identities for products of sines and cosines expressed as sums and differences
- convert sums $a\cos x+b\sin x$ to $R\cos(x\pm\alpha)$ or $R\sin(x\pm\alpha)$ and apply these to sketch graphs, solve equations of the form $a\cos x+b\sin x=c$ and solve problems
- prove and apply other trigonometric identities such as $\cos 3x=4\cos^3 x-3\cos x$

Applications of trigonometric functions to model periodic phenomena involving simple harmonic motion such as heights of tides and other forms of wave motion:

- model periodic motion using sine and cosine functions and understand the relevance of the period and amplitude of these functions in the model

Topic 2: Matrices

Matrix arithmetic:

- matrix definition and notation
- addition and subtraction of matrices, scalar multiplication, matrix multiplication, multiplicative identity and inverse
- calculate the determinant and inverse of 2×2 matrices
- solve matrix equations of the form $AX=B$, where A is a 2×2 matrix and X and B are column vectors

Simultaneous equations:

- solve simultaneous equations in two variables using matrices
- interpret geometrically cases where there is a unique solution, no solution or infinitely many solutions.

Transformations in the plane:

- translations and their representation as column vectors
- basic linear transformations: dilations of the form $(x, y) \longrightarrow (\lambda_1 x, \lambda_2 y)$, rotations about the origin and reflection in a line which passes through the origin, and the representations of these transformations by 2×2 matrices
- application of these transformations to points in the plane and geometric objects
- composition of linear transformations and the corresponding matrix products
- inverses of linear transformations and the relationship with the matrix inverse
- the relationship between the determinant and the effect of a linear transformation on area
- geometric results by matrix multiplications; for example: show that the combined effect of 2 reflections is a rotation

Topic 3: Real and complex numbers

Proofs involving numbers. For example:

- There are infinitely many prime numbers.
- An integer n is odd if and only if n^2 is odd.
- The sum of two consecutive odd integers is divisible by 4.
- If n and m are even integers then $n + m$ is an even integer.
- For all real numbers x and y, if x is rational and y is irrational, then $x + y$ is irrational.
- For all non-zero real numbers x and y, we have $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$.

Infinite decimal expansions:

- express rational numbers as terminating or eventually recurring decimals and vice versa
- characterise the decimal expansion of irrational numbers

Existence of irrationals:

- direct construction of decimals such as $0.0100100010\ldots$
- proofs of irrationality by contradiction for numbers such as $\sqrt{2}$ and $\log_2 5$

Complex numbers:

- the imaginary number i as a root of the equation $x^2 = -1$
- complex numbers in the form; $a + bi$ where a and b are real numbers
- real and imaginary parts of a complex number
- complex conjugates
- complex number arithmetic: addition, subtraction, multiplication and division

The complex plane:

- complex numbers as points in a plane with real and imaginary parts as Cartesian coordinates
- addition of complex numbers as vector addition in the complex plane
- location of complex conjugates in the complex plane

Roots of equations:

- the general solution of real quadratic equations
- complex conjugate solutions of real quadratic equations
- linear factors of real quadratic polynomials

Topic 4: Graph Theory

Basic graph concepts:

- examples to illustrate a broad range of applications of graphs as models including molecular structure, electrical circuits, social networks, connection of utilities, braced rectangular frameworks
- use of examples to discuss types of problems to be met in graph theory including existence problems, construction problems, counting problems and optimisation problems
- vertices, edges, loops, and multiple edges
- the degree of a vertex and the result that the sum of all the vertex degrees is equal to twice the number of edges (handshaking lemma)
- simple graphs, subgraphs, connectedness, components and complements including recognising subgraphs of a given graph, determining if a graph is connected or not, finding the complement of a graph
- isomorphisms of graphs and finding simple criteria to show two graphs are not isomorphic
- simple graphs including all simple graphs with four or fewer vertices and, in particular the simple connected graphs with four or fewer vertices
- regular graphs including the Platonic graphs and proving results such as for a regular graph with n vertices, each having degree r, there are $\frac{nr}{2}$ edges

- the complete graph K_n on n vertices. This includes the proof of the result that the complete graph K_n has $\frac{n(n-1)}{2}$ edges and applications of this rule
- bipartite graphs, and the complete bipartite graph $K_{m,n}$ on m and n vertices. Includes the result that a graph is bipartite if and only if every cycle of the graph has an even number of edges

Trees:

- examples of trees including oil or gas distribution networks and chemical bonding
- equivalent conditions for a simple graph with n vertices to be a **tree** including 'the graph is connected and has $n-1$ edges' and 'any two vertices are connected by exactly one path'. Problems include questions such as 'How many trees can be formed with five vertices?'
- construct spanning trees from given connected graphs

Planarity:

- define planarity and discuss examples of planar graphs such as in planning a golf course and circuit boards
- show the planarity of various types of graphs, including all trees, K_n, the complete graph with n vertices, if $n \leq 4$, and $K_{m,n}$ the complete bipartite graph with m and n vertices, if $m \leq 2$ or $n \leq 2$
- planar graphs corresponding to polyhedra
- prove and apply Euler's formula $v + f - e = 2$ for simple connected planar graphs
- use Euler's formula to establish the following inequalities for simple connected planar graphs with at least three vertices
- $e \leq 3v - 6$ and
- $e \leq 2v - 4$ for graphs without triangles
- the utilities problem and the use of Euler's formula to establish the non-planarity of K_5 and $K_{3,3}$
- regular polyhedra and the proof of uniqueness of the Platonic solids via Euler's formula

Closed paths:

- the Königsberg bridge problem
- the concepts of Euler circuits and Euler trails, and the necessary and sufficient conditions for the existence of Euler trails and Euler circuits
- Fleury's algorithm for constructing Euler trails
- the concept of Hamiltonian cycles and paths

Achievement Standards Units 1 and 2

	Concepts and Techniques	Reasoning and Communication
A	The student: • understands and applies concepts and techniques in recurrence relations, combinatorics, geometry and vectors in the plane, matrices, trigonometry, the real and complex numbers and graph theory to solve a wide range of problems including non-routine problems • uses digital technologies appropriately and skilfully to solve non-routine problems, and to display and organise information effectively • represents varied mathematical information accurately, precisely and effectively in numerical, graphical and symbolic form • translates efficiently and effectively between practical problems and their mathematical model in a variety of situations including unfamiliar contexts	The student: • synthesises mathematical techniques, results and ideas creatively to solve problems • constructs proofs in a variety of settings using a wide range of techniques and determines the solutions to problems, that require the application of multi-step mathematical reasoning • analyses and interprets the reasonableness of results and solutions to a wide range of problem types • analyses and interprets results with comprehensive consideration of the validity and limitations of the use of any models • communicates observations, judgments and decisions which are succinct, clear, reasoned, and evidenced, as needed • demonstrates and communicates an understanding of the inter-relatedness of different representations of mathematical information
B	The student: • understands and applies most concepts and techniques in recurrence relations, combinatorics, geometry and vectors in the plane, matrices, trigonometry, the real and complex numbers and graph theory to solve a wide range of problems • uses digital technologies appropriately to solve non-routine problems, and to display and organise information effectively • represents varied mathematical information accurately in numerical, graphical and symbolic form • translates appropriately between practical problems and their mathematical model in familiar situations	The student: • solves problems that require the interpretation of mathematical information • constructs proofs in familiar situations • analyses the reasonableness of results and solutions to problems • analyses results with comprehensive consideration of the validity and limitations of the use of any models • demonstrates and communicates reasoned observations and decisions • demonstrates an understanding of the inter-relatedness of different representations of mathematical information
C	The student: • understands and applies some concepts and techniques in recurrence relations, combinatorics, geometry and vectors in the plane, matrices, trigonometry, the real and complex numbers and graph theory to solve familiar problems • uses digital technologies appropriately to solve standard problems and to display and organise information	The student: • solves familiar problems that require the interpretation of mathematical information • reproduces and adapts previously seen proofs • analyses the reasonableness of results and solutions to familiar problems • analyses results with consideration of the validity and limitations of the use of any models

	Concepts and Techniques	Reasoning and Communication
	• represents familiar mathematical and statistical information in numerical, graphical and symbolic form • solves practical problems based on standard techniques and models	• demonstrates and communicates observations • recognises the inter-relatedness of different representations of mathematical information
D	The student: • demonstrates limited understanding and application of some concepts and techniques in recurrence relations, combinatorics, geometry and vectors in the plane, matrices, trigonometry, the real and complex numbers and graph theory to solve standard problems • uses digital technologies to undertake simple calculations and to display and organise information • represents some familiar mathematical and statistical information in numerical, graphical and symbolic form • applies concepts and techniques to solve routine problems set in a practical context	The student: • solves routine problems that require the interpretation of mathematical information • reproduces some seen proofs • recognises the reasonableness of results and solutions to routine problems • communicates observations • recognises the inter-relatedness of some representations of mathematical information
E	The student: • demonstrates limited familiarity in recurrence relations, combinatorics, geometry and vectors in the plane, matrices, trigonometry, the real and complex numbers and graph theory • uses digital technologies to undertake routine calculations to solve familiar problems • represents mathematical and statistical information in limited forms • solves some routine problems set in a practical context	The student: • recognises the solution to routine problems • communicates some observations • recognises the representations of mathematical information

Unit 3

Unit description

Unit 3 of Specialist Mathematics contains four topics, 'Vectors'; 'Matrices and systems of equations'; 'Complex numbers' and 'Functions and calculus'. In Units 1 and 2, the study of matrices and vectors was begun. In Unit 1 vectors in 2-dimensional space were the focus. In this unit, 3-dimensional vectors are studied, and vector equations and vector calculus are introduced, with the latter extending students' knowledge of calculus from Mathematical Methods. Cartesian and vector equations of lines, together with equations of planes enable students to work with geometric considerations as well as motion in 3-dimensional space.

In Unit 2, the emphasis for matrices was on the solution of simultaneous equations in two variables and transformations of the plane. In Unit 3, the emphasis is on the solution of systems of equations with three variables and the geometric interpretation of these solutions.

The Cartesian form of complex numbers was introduced in Unit 2 and the study of complex numbers is now extended to the polar form.

The study of functions and techniques of calculus begun in Mathematical Methods are extended and utilised in the sketching of graphs and the solution of problems involving integration.

Learning Outcomes

By the end of this unit, students will:

- understand the concepts and techniques in vectors, matrices, complex numbers, functions and calculus
- apply reasoning skills and solve problems in vectors, matrices, complex numbers, functions and calculus
- communicate their arguments and strategies when solving problems
- construct proofs of results
- interpret mathematical information and ascertain the reasonableness of their solutions to problems.

Content descriptions

Topic 1: Vectors in three dimensions

The algebra of vectors in three dimensions:

- review the concepts of vectors from Unit 1 and extend to three dimensions including introducing the unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$

Vector equations:

- use vector equations of curves in two or three dimensions using a parameter and determine a 'corresponding' Cartesian equation in the two-dimensional case
- determine a vector equation of a straight line, given the position of two points or equivalent information, in both two and three dimensions
- the position of two particles each described as a vector function of time and determining if their paths cross or if the particles meet
- use the cross product to determine a vector normal to a given plane.
- determine vector and Cartesian equations of a plane

Vector calculus:

- position vector as a function of time
- derive the Cartesian equation of a path given as a vector equation in two dimensions
- differentiate and integrate a vector function with respect to time
- determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration
- apply vector calculus to motion in two and three-dimensions

Topic 2: Matrices and systems of equations

Systems of linear equations:

- the general form of a system of linear equations in several variables
- use elementary techniques of elimination to solve a system of linear equations
- three cases for solutions of systems of equations: a unique solution, no solution and infinitely many solutions
- the role of the matrix inverse in the solution of equations which have a unique solution
- geometric interpretation of a solution of a system of equations with three variables

Row reduction methods:

- the augmented matrix as an efficient way of representing a system of linear equations
- row reduction of the augmented matrix as an equivalent of systematic elimination of variables in equations
- use of an augmented matrix to calculate the inverse of a matrix
- row echelon form

Topic 3: Complex Numbers

Cartesian forms including a review of:

- real and imaginary parts $Re(z)$ and $Im(z)$ of a complex number z
- Cartesian form
- complex arithmetic using Cartesian forms

Complex arithmetic using polar form:

- the modulus $|z|$ of a complex number z
- the argument $Arg\ (z)$ of a non-zero complex number z
- the polar coordinates of a number in the complex plane
- conversions between Cartesian and polar form
- multiplication, division, and powers of complex numbers in polar form and the geometric interpretation of these
- De Moivre's theorem

The complex plane (The Argand plane):

- addition of complex numbers as vector addition in the complex plane
- multiplication as a linear transformation in the complex plane
- identification of subsets of the complex plane determined by relations such as $|z-3i| \leq 4$, $\frac{\pi}{4} \leq Arg(z) \leq \frac{3\pi}{4}$, $Re(z) > Im(z)$, and $|z-1| = 2|z-i|$

Roots of complex numbers:

- the n^{th} roots of unity and their location on the unit circle
- the n^{th} roots of complex numbers and their location in the complex plane

Factorisation of polynomials:

- prove and apply the factor theorem and the remainder theorem for polynomials
- conjugate roots for polynomials with real coefficients
- solution of simple polynomial equations

Topic 4: Functions and calculus

Functions:

- determine when the composition of two functions is defined
- find the composition of two functions
- determine if a function is one-to-one
- find the inverse function of a one-to-one function
- the reflection property of the graphs of a function and its inverse

Sketching graphs:

- the relationship between the graph of $y = f(x)$ and the graphs of $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$
- sketching the graphs of simple rational functions where the numerator and denominator are polynomials of low degree.

Integration techniques:

- integration using the trigonometric identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and $1 + \tan^2 x = \sec^2 x$
- use of substitution $u = g(x)$ to integrate expressions of the form $f\left(g(x)\right)g'(x)$
- inverse trigonometric functions arcsine, arccosine and arctangent
- the derivative of the inverse trigonometric functions, arcsine, arccosine and arctangent
- integrating expressions of the form $\frac{\pm 1}{\sqrt{a^2 - x^2}}$ and $\frac{a}{a^2 + x^2}$

Applications of integral calculus:

- the calculation of areas between curves determined by functions
- volumes of solids of revolution about either axis
- numerical integration using technology

Unit 4

Unit description

Unit 4 contains two topics, 'Further calculus and applications of calculus' and 'Statistical inference for bivariate continuous data'.

In Unit 4, the study of differentiation and integration of functions is continued and the techniques developed from this and previous topics in calculus are applied in the area of simple differential equations and in particular in biology and kinematics. These topics serve to demonstrate the applicability of the mathematics learnt throughout this subject.

In this unit all of the students' previous experience in statistics is drawn together in the study of statistical inference for bivariate continuous data. This is a topic that demonstrates the utility and power of statistics.

Learning Outcomes

By the end of this unit, students:

- understand the concepts and techniques in applications of calculus and statistical inference
- apply reasoning skills and solve problems in applications of calculus and statistical inference
- communicate their arguments and strategies when solving problems
- construct proofs of results
- interpret mathematical information and ascertain the reasonableness of their solutions to problems.

Content descriptions

Topic 1: Further calculus and applications of calculus

Further Integration techniques:

- partial fractions
- use partial fractions where necessary for integration in simple cases
- integration by parts

Differential equations:

- solve simple first order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and in general differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables
- slope (direction or gradient) fields of a first order differential equation
- formulate differential equations including the logistic equation that will arise for example, in chemistry, biology and economics in situations where rates are involved

Modelling motion:

- momentum, force, resultant force, action and reaction
- constant and non-constant force
- motion of a body under concurrent forces
- motion in a straight line with both constant and non-constant acceleration including the use of expressions $\frac{dv}{dt}$, $v\frac{dv}{dx}$ and $\frac{d(\frac{1}{2}v^2)}{dx}$ for acceleration

Topic 2: Statistical inference for continuous data

Data investigation for inference about a mean difference:

- by considering a variety of (matched) paired data contexts, including before- and - after, use a single sample of differences of paired observations and the mean of these differences to consider evidence for significant change
- use a confidence interval to estimate the mean difference, identifying the statistical population or general situation for which the inference is appropriate, and provide simple justifications for the assumption of independent and identically distributed observations or observed differences
- for paired data, use binomial probabilities to find the chance of obtaining at least the observed number of positive (or negative) differences assuming these differences are equally-likely
- use this probability to comment on the data of differences in context

- the concept of a small chance of obtaining data under an assumption providing evidence against the assumption

Investigating data on two possibly related continuous random variables:

- decide which is the explanatory variable and which is the response variable, if such a relationship exists
- use a scatterplot for an initial indication of whether a straight line might be an appropriate model to fit
- use digital technologies to fit a straight line to the data using the method of least squares
- the concept of least squares estimation
- the slope and intercept of the fitted line are (point) estimates of parameters in a model. (This model assumes that the mean of the response variable is a straight line in the explanatory variable.)
- the correlation coefficient as a measure of the strength of the linear relationship between two variables
- residual plots and their use in assessing the appropriateness of fitting a linear model and identifying unusual observations
- interpret the estimated slope in terms of changes in the expected value of the response variable as the explanatory variable changes
- use digital technologies to carry out simple permutations of the response values in the collected data and calculate slopes to estimate the probability of obtaining the original observed slope or greater (in absolute value) assuming there is no linear relationship between explanatory and the response variable
- use this estimated probability to comment on the data with respect to the original scatterplot
- for data for which a straight line model may not be appropriate, use transformations of the data, such as logarithmic or power functions, and the above procedures to investigate the possibility of fitting a linear model to transformed data

Critical analysis of reports showing mean differences, correlations or other statistics for bivariate data:

- check, where possible, the quality of the data and the procedures for obtaining the data
- comment on issues relating to appropriateness of procedures and provision of evidence for conclusions

AUSTRALIAN CURRICULUM, ASSESSMENT AND REPORTING AUTHORITY

Achievement Standards Units 3 and 4

	Concepts and Techniques	Reasoning and Communication
A	The student: • understands and applies concepts and techniques in vectors, complex numbers, functions, calculus and statistics to solve a wide range of problems including non-routine problems • uses digital technologies appropriately and skilfully to solve problems, and to display and organise information • represents mathematical and statistical information accurately, precisely and effectively in numerical, graphical and symbolic form • uses algebraic, geometric, statistical and modelling skills and multi-step logic to solve varied problems • translates efficiently and effectively between practical problems and their mathematical or statistical model in a variety of situations including unfamiliar contexts	The student: • synthesises mathematical and statistical techniques, results and ideas creatively to solve problems • constructs proofs in a variety of settings using a range of techniques and determines the solutions to problems, that require the application of multi-step mathematical reasoning • analyses and interprets the reasonableness of results and solutions to a wide range of problem types • analyses and interprets results with comprehensive consideration of the validity and limitations of the use of any models • communicates observations, judgments and decisions which are succinct, clear, reasoned, and evidenced, as needed. • demonstrates and communicates an understanding of the inter-relatedness of different representations of mathematical and statistical information
B	The student: • understands and applies most concepts and techniques in vectors, complex numbers, functions, calculus and statistics • uses digital technologies appropriately and competently to solve problems and to display and organise information • represents mathematical and statistical information accurately in numerical, graphical and symbolic form • uses algebraic, geometric, statistical and modelling skills and multi-step logic to solve problems • translates efficiently and effectively between practical problems and their mathematical or statistical model in a variety of situations	The student: • solves problems that require the interpretation of mathematical and statistical information • constructs proofs in familiar situations • analyses the reasonableness of results and solutions to mathematical and statistical problems • demonstrates and communicates reasoned observations and decisions • analyses results with consideration of the validity and limitations of the use of any models • demonstrates an understanding of the inter-relatedness of different representations of mathematical and statistical information
C	The student: • understands and applies some concepts and techniques in vectors, complex numbers, functions, calculus and statistics • uses digital technologies appropriately to solve problems to display and organise information • represents mathematical and statistical information in numerical, graphical and symbolic form • accurately applies concepts and techniques to solve familiar problems • determines the solution to familiar problems requiring the application of limited combinations of concepts and techniques	The student: • solves familiar problems that require the interpretation of mathematical and statistical information • reproduces and adapts previously seen proofs • analyses the reasonableness of results and solutions to familiar problems • demonstrates and communicates observations • analyses results from standard problems with consideration of the validity and limitations of the use of any mathematical or statistical models • recognises the inter-relatedness of different representations of mathematical and statistical information

	Concepts and Techniques	Reasoning and Communication
D	The student: • demonstrates limited understanding and application of some concepts and techniques in vectors, complex numbers, functions, calculus and statistics • uses digital technologies to solve some problems • represents mathematical and statistical information in limited forms • applies concepts and techniques to solve routine problems	The student: • solves routine problems that require the interpretation of mathematical and statistical information • reproduces some seen proofs • recognises the reasonableness of results and solutions to routine problems • communicates observations • recognises the inter-relatedness of some representations of mathematical and statistical information
E	The student: • demonstrates limited familiarity in vectors, complex numbers, functions, calculus and statistics • follows procedures to solve simple problems • uses digital technologies to represent information and to solve simple problems	The student: • recognises the solution to routine problems • communicates some observations • recognises the representations of mathematical and statistical information

GLOSSARY ITEMS

Unit 1

Recurrence Relations

Arithmetic sequence

An arithmetic sequence is a sequence of numbers such that the difference of any two successive members of the sequence is a constant. For instance, the sequence

2, 5, 8, 11, 14, 17, …

is an arithmetic sequence with common difference 3.

If the initial term of an arithmetic sequence is a and the common difference of successive members is d, then the nth term t_n, of the sequence, is given by:

$t_n = a + (n - 1)d$ for $n \geq 1$

A recursive definition is

$t_1 = a$, $t_n = t_{n-1} + d$ where d is the common difference and $n \geq 2$.

Factorial sequence

The factorial sequence is:

1, 1, 2, 6, 24, 120,…

$t_n = n! = n(n - 1)(n - 2)\ldots 3 \times 2 \times 1$

The sequence is defined by the recurrence relation

$0! = 1$, $n! = n(n - 1)!$ for $n \geq 1$.

Fibonacci sequence

The Fibonacci numbers are the numbers in the following integer sequence:

1, 1, 2, 3, 5, 8, 13,…..

The sequence is called the Fibonacci sequence.

The first two Fibonacci numbers are 1 and 1, and each subsequent number is the sum of the previous two.

The sequence F_n of Fibonacci numbers is defined by the recurrence relation

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

with initial values $F_1 = 1$ and $F_2 = 1$.

First order linear recurrence relation

A first order linear recurrence relation is defined by the equation

$t_n = bt_{n-1} + c$ for $n \geq 2$ and $t_1 = a$.

Geometric sequence

A geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed number called the **common ratio**. For example, the sequence

3, 6, 12, 24, ...

is a geometric sequence with common ratio 2. Similarly the sequence

40, 20, 10, 5, 2.5, …

is a geometric sequence with common ratio $\frac{1}{2}$.

If the initial term of a geometric sequence is a and the common ratio of successive members is r, then the nth term t_n, of the sequence, is given by:

$t_n = ar^{n-1}$ for $n \geq 1$

A recursive definition is

$t_1 = a, \quad t_n = rt_{n-1}$ for $n \geq 2$ and where r is the constant ratio

Partial sums of a sequence

The sequence of partial sums of a sequence $t_1, \ldots, t_s, \ldots$ is defined by

$S_n = t_1 + \ldots + t_n$

Partial sum of an arithmetic sequence

The partial sum S_n of the first n terms of an arithmetic sequence with first term a and common difference d.

$a, a + d, a + 2d, \ldots\ldots, a + (n-1)d, \ldots$

is

$S_n = \frac{n}{2}(a + t_n) = \frac{n}{2}(2a + (n-1)d)$ where t_n is the n^{th} term of the sequence.

The partial sums form a sequence with $S_n = S_{n-1} + t_n$ and $S_1 = t_1$

Partial sums of a geometric sequence

The partial sum S_n of the first n terms of a geometric sequence with first term a and common ratio r,

$a, ar, ar^2, \ldots\ldots, ar^{n-1}, \ldots$

is

$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1.$

The partial sums form a sequence with $S_n = S_{n-1} + t_n$ and $S_1 = t_1$.

Principle of mathematical induction

Let there be associated with each positive integer n, a proposition $P(n)$.

If

1. $P(1)$ is true, and
2. for all k, $P(k)$ is true implies $P(k+1)$ is true,

then $P(n)$ is true for all positive integers n.

Combinatorics

Arranging n objects in an ordered list

The number of ways to arrange n different objects in an ordered list is given by

$n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1 = n!$

Combinations (Selections)

The number of selections of n objects taken r at a time (that is, the number of ways of selecting r objects out of n) is denoted by $^nC_r = \binom{n}{r}$ and is equal to

$$\frac{n!}{r!(n-r)!}$$

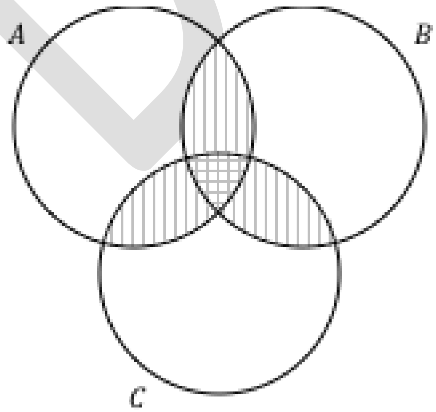
Factorial (See **factorial sequence** in recurrence relations)

Inclusion – exclusion principle

- Suppose A and B are subsets of a finite set X then

  $|A \cup B| = |A| + |B| - |A \cap B|$

- Suppose A, B and C are subsets of a finite set X then



$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

This result can be generalised to 4 or more sets.

Multiplication principle

Suppose a choice is to be made in two stages. If there are a choices for the first stage and b choices for the second stage, no matter what choice has been made at the first stage, then there are ab choices altogether. If the choice is to be made in n stages and if for each i, there are a_i choices for the i^{th} stage then there are $a_1 a_2 \ldots a_n$ choices altogether.

Pascal's triangle

Pascal's triangle is an arrangement of numbers. In general the n^{th} row consists of the binomial coefficients ${}^nC_r$ or $\binom{n}{r}$ with the $r = 0, 1, \ldots, n$

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

In Pascal's triangle any term is the sum of the two terms 'above' it.

For example 10 = 4 + 6.

Identities include:

- The recurrence relation, ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$
- ${}^nC_k = \frac{n}{k}\,{}^{n-1}C_{k-1}$

Permutations

A permutation of n objects is an arrangement or rearrangement of n objects (order is important). The number of arrangements of n objects is $n!$ The number of permutations of n objects taken r at a time is denoted ${}^nP_r$ and is equal to

$$n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}.$$

Pigeon-hole principle

If there are n pigeon holes and $n + 1$ pigeons to go into them, then at least one pigeon hole must get 2 or more pigeons.

Geometry

Glossary for Proof

Contradiction-Proof by

Assume the opposite (**negation**) of what you are trying to prove. Then proceed through a logical chain of argument till you reach a demonstrably false conclusion. Since all the reasoning is correct and a false conclusion has been reached the only thing that could be wrong is the initial assumption. Therefore the original statement is true.

For example: the result $\sqrt{2}$ is irrational can be proved in this way by first assuming $\sqrt{2}$ is rational.

The following are examples of results that are often proved by contradiction:

- If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.
- If an interval (line segment) subtends equal angles at two points on the same side of the interval (line segment), then the two points and the endpoints of the interval are concyclic.

Implication and Converse

Implication: if P then Q Symbol: $P \Rightarrow Q$

Examples:

- If a quadrilateral is a rectangle then the diagonals are of equal length and they bisect each other.
- If $x = 2$ then $x^2 = 4$.
- If an animal is a kangaroo then it is a marsupial.
- If a quadrilateral is cyclic then the opposite angles are supplementary.

Converse of a statement The converse of the statement 'If P then Q' is 'If Q then P' Symbolically the converse of $P \Rightarrow Q$ is: $Q \Rightarrow P$ or $P \Leftarrow Q$

The converse of a true statement need not be true.

Examples:

- **Statement:** If a quadrilateral is a rectangle then the diagonals are of equal length and they bisect each other.

  Converse statement: If the diagonals of a quadrilateral are of equal length and bisect each other then the quadrilateral is a rectangle. (In this case the converse is true.)

- **Statement:** If $x = 2$ then $x^2 = 4$.

  Converse statement: If $x^2 = 4$ then $x = 2$. (In this case the converse is false.)

- **Statement:** If an animal is a kangaroo then it is a marsupial.

Converse statement: If an animal is a marsupial then it is a kangaroo. (In this case the converse is false.)

- **Statement:** If a quadrilateral is cyclic then the opposite angles are supplementary.

  Converse statement: If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic. (In this case the converse is true.)

Contrapositive

The contrapositive of the statement 'If P then Q' is 'If not Q then not P'. The contrapositive of a true statement is also true. (not Q is the **negation** of the statement Q)

Examples:

- **Statement:** A rectangle is a quadrilateral that has diagonals of equal length and the diagonals bisect each other.

  Contrapositive: If the diagonals of a quadrilateral are not of equal length or do not bisect each other then the quadrilateral is not a rectangle.

- **Statement:** If $x = 2$ then $x^2 = 4$.

  Contrapositive: If $x^2 \neq 4$ then $x \neq 2$.

- **Statement:** A kangaroo is a marsupial.

  Contrapositive: If an animal is not a marsupial then it is not a kangaroo.

- **Statement:** The opposite angles of a cyclic quadrilateral are supplementary

  Contrapositive: If the opposite angles of quadrilateral are not supplementary then the quadrilateral is not cyclic.

Counterexample

A Counterexample is an example that demonstrates that a statement is not true.

Examples:

- **Statement**: If $x^2 = 4$ then $x = 2$.

  Counterexample: $x = -2$ provides a counterexample.

- **Statement**: If the diagonals of a quadrilateral intersect at right angles then the quadrilateral is a rhombus.

  Counterexample: A kite with the diagonals not bisecting each other is not a rhombus. Such a kite provides a counterexample to the statement. The diagonals of a kite do intersect at right angles.

- **Statement**: Every convex quadrilateral is a cyclic quadrilateral.

  Counterexample: A parallelogram that is not a rectangle is convex, but not cyclic.

Equivalent statements

Statements P and Q are equivalent if P $\Rightarrow$ Q and Q $\Rightarrow$ P. The symbol $\Leftrightarrow$ is used. It is also written as P if and only if Q or P iff Q.

Examples:

- A quadrilateral is a rectangle if and only if the diagonals of the quadrilateral are of equal length and bisect each other.
- A quadrilateral is cyclic if and only if opposite angles are supplementary.

Negation

If P is a statement then the statement 'not P', denoted by $\neg$P is the negation of P. If P is the statement 'It is snowing.' then $\neg$P is the statement 'It is not snowing.'.

Quantifiers

For all (For each)

Symbol $\forall$

- For all real numbers $x, x^2 \geq 0.$ ($\forall$ real numbers $x, x^2 \geq 0.$)
- For all triangles the sum of the interior angles is 180° .($\forall$ triangles the sum of the interior angles is 180°.)
- For each diameter of a given circle each angle subtended at the circumference by that diameter is a right angle.

There exists

Symbol $\exists$

- There exists a real number that is not positive ($\exists$ a real number that is not positive.)
- There exists a prime number that is not odd. ($\exists$ a prime number that is not odd.)
- There exists a natural number that is less than 6 and greater than 3.
- There exists an isosceles triangle that is not equilateral.

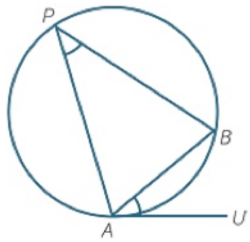
The quantifiers can be used together.

For example: $\forall\, x \geq 0, \exists\, y \geq 0$ such that $y^2 = x$.

Glossary of Geometric Terms and Listing of Important Theorems

Alternate segment

The word 'alternate' means 'other'. The chord *AB* divides the circle into two segments and *AU* is tangent to the circle. Angle *APB* 'lies in' the segment on the other side of chord *AB* from angle *BAU.* We say that it is in the **alternate segment**.

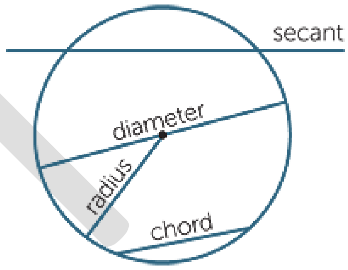


Cyclic quadrilateral

A **cyclic quadrilateral** is a quadrilateral whose vertices all lie on a circle.

Lines and line segments associated with circles

Any line segment joining a point on the circle to the centre is called a **radius**. By the definition of a circle, any two radii have the same length called the radius of the circle. Notice that the word 'radius' is used to refer both to these intervals and to the common length of these intervals.



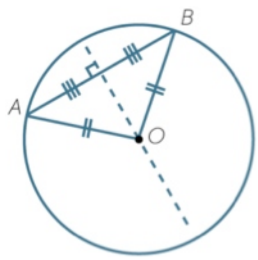
- An interval joining two points on the circle is called a **chord**.
- A chord that passes through the centre is called a **diameter**. Since a diameter consists of two radii joined at their endpoints, every diameter has length equal to twice the radius. The word 'diameter' is use to refer both to these intervals and to their common length.

A line that cuts a circle at two distinct points is called a **secant**. Thus a chord is the interval that the circle cuts off a secant, and a diameter is the interval cut off by a secant passing through the centre of a circle.

Circle Theorems

Result 1

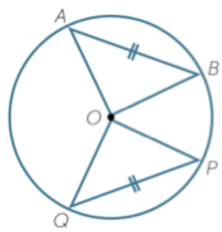
Let *AB* be a chord of a circle with centre *O*.



The following three lines coincide:

- The bisector of the angle $\angle AOB$ subtended at the centre by the chord.
- The line segment (interval) joining *O* and the midpoint of the chord *AB*.
- The perpendicular bisector of the chord *AB*.

Result 2



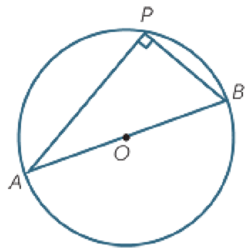
- Equal chords of a circle subtend equal angles at the centre.

In the diagram shown $\angle AOB = \angle POQ$.

Result 3

- An angle in a semicircle is a right angle.

Let *AOB* be a diameter of a circle with centre *O*, and let *P* be any other point on the circle. The angle ∠*APB* subtended at *P* by the diameter *AB* is called an **angle in a semicircle.**

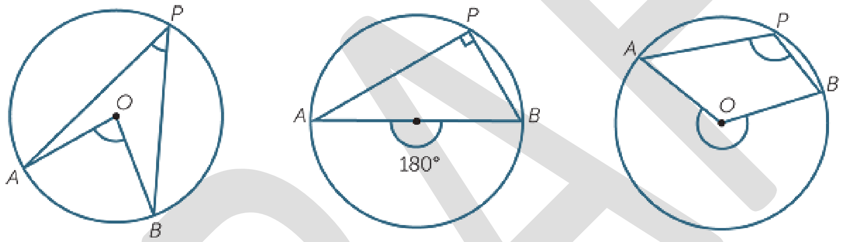


Converse

- The circle whose diameter is the hypotenuse of a right-angled triangle passes through all three vertices of the triangle.

Result 4

- An angle at the circumference of a circle is half the angle subtended at the centre by the same arc. In the diagram shown ∠*AOB* = 2∠*APB*



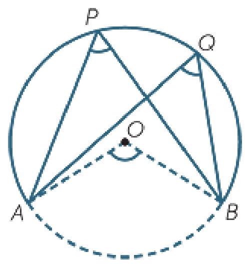
The arc *AB* **subtends** the angle ∠*AOB* at the centre. The arc also subtends the angle ∠*APB*, called an **angle at the circumference** subtended by the arc *AB*.

Result 5

- Two angles at the circumference subtended by the same arc are equal.

∠APB = ∠AQB

In the diagram, the two angles ∠*APB* and ∠*AQB* are subtended by the same arc *AB*.



Result 6

- The opposite angles of a cyclic quadrilateral are supplementary.

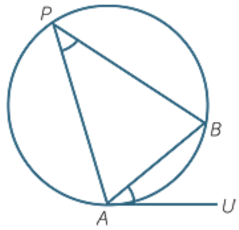
Converse

- If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Result 7

Alternate segment theorem

An angle between a chord and a tangent is equal to any angle in the alternate segment.

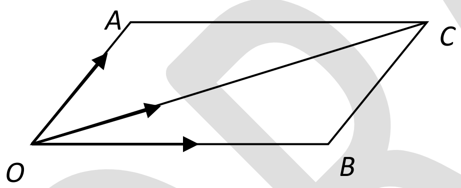


In the diagram $\angle BAU = \angle APB$.

Vectors in the plane

Addition of vectors (see Vector for definition and notation)

Given vectors **a** and **b** let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be directed line segments that represent **a** and **b.** They have the same initial point O. The sum of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is the directed line segment $\overrightarrow{OC}$ where C is a point such that $OACB$ is a parallelogram. This is known as the **parallelogram rule**.



If **a** = (a_1, a_2) and **b** = (b_1, b_2) then **a** + **b** = $(a_1 + b_1, a_2 + b_2)$

In component form if **a** = a_1**i** + a_2**j** and **b** = b_1**i** + b_2**j** then

a + **b** = $(a_1 + b_1)$**i** + $(a_2 + b_2)$**j**

Properties of vector addition:

- **a** + **b** = **b** + **a** (commutative law)
- (**a** + **b**) + **c** = **a** + (**b** + **c**) (associative law)
- **a** + **0** = **0** + **a** = **a**
- **a** + (–**a**) = **0**

Magnitude of a vector (see Vector for definition and notation)

The magnitude of a vector **a** is the length of any directed line segment that represents **a.** It is denoted by |**a**|.

Multiplication by a scalar

Let **a** be a non-zero vector and k a positive real number (scalar) then the scalar multiple of **a** by k is the vector $k\mathbf{a}$ which has magnitude $|k|$ |**a**| and the same direction as **a.** If k is a negative real number, then k **a** has magnitude $|k|$ |**a**| and but is directed in the opposite direction to **a**. (see **negative of a vector**)

Some properties of scalar multiplication are:

- k (**a** + **b**) = k **a** + k **b**
- $h(k$ (**a**)) = (hk)**a**
- 1**a** = **a**

Negative of a vector (see Vector for definition and notation)

Given a vector **a**, let $\overrightarrow{AB}$ be a directed line segment representing **a.** The negative of **a**, denoted by –**a,** is the vector represented by $\overrightarrow{BA}$. The following are properties of vectors involving negatives:

- **a** + (–**a) =** (–**a**) + **a** = **0**
- –(–**a**) = **a**

Scalar product (see Vector for definition and notation)

a = (a_1, a_2) and **b =** (b_1, b_2) then the scalar product **a**.**b** is the real number

$a_1 b_1 + a_2 b_2$. The geometrical interpretation of this number is **a.b** =|**a**| |**b**| cos (θ) where θ is the angle 'between' **a** and **b**

When expressed in **i**, **j**, notation, if **a** = a_1**i** + a_2**j** and **b** = b_1**i** + b_2**j** then

a.b = $a_1 b_1 + a_2 b_2$

Note |**a**| = $\sqrt{\mathbf{a.a}}$

Subtraction of vectors (see Vector for definition and notation)

a – b = a + (–b)

Unit vector (see Vector for definition and notation)

A unit vector is a vector with magnitude 1. Given a vector **a** the unit vector in the same direction as **a** is $\frac{1}{|\mathbf{a}|}$ **a.** This vector is often denoted as $\hat{\mathbf{a}}$.

Vector projection (see Vector for definition and notation)

Let **a** and **b** be two vectors and write θ for the angle between them. The projection of a vector **a** on a vector **b** is the vector

|**a**| cos θ $\hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b.**

The projection of a vector **a** on a vector **b** is $(\mathbf{a}.\hat{\mathbf{b}})\,\hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b.**

This projection is also given by the formula $\dfrac{\mathbf{a}.\mathbf{b}}{\mathbf{b}.\mathbf{b}}\,\mathbf{b}$.

Vector

In Physics the name vector is used to describe a physical quantity like velocity or force that has a magnitude and direction.

A vector is an entity **a** which has a given length (magnitude) and a given direction. If $\overrightarrow{AB}$ is a directed line segment with this length and direction, then we say that $\overrightarrow{AB}$ represents **a.**

If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector, they are parallel and have the same length.

The **zero vector** is the vector with length zero.

In two dimensions, every vector can be represented by a directed line segment which begins at the origin. For example, the vector $\overrightarrow{BC}$ from B(1,2) to C(5,7) can be represented by the directed line segment $\overrightarrow{OA}$, where A is the point (4,5). The **ordered pair** notation for a vector uses the co-ordinates of the end point of this directed line segment beginning at the origin to denote the vector, so $\overrightarrow{BC}$ = (4,5) in ordered pair notation. The same vector can be represented in **column vector** notation as $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Unit 2

Trigonometry

Angle sum and difference identities

$\sin(A + B) = \sin A \cos B + \sin B \cos A$

$\sin(A - B) = \sin A \cos B - \sin B \cos A$

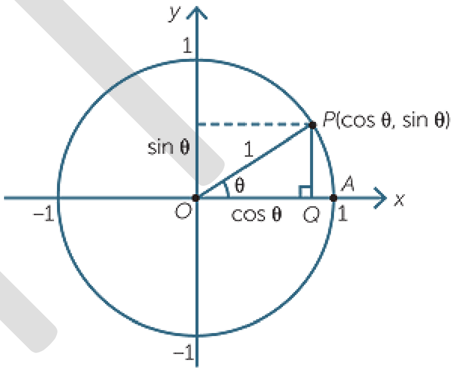
$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

Cosine and sine functions

Since each angle θ measured anticlockwise from the positive *x*-axis determines a point P on the unit circle, we will define

- the cosine of θ to be the x-coordinate of the point *P*
- the sine of θ to be the y-coordinate of the point *P*
- the tangent of θ is the gradient of the line segment *OP*



Double angle formula

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$

  $= 2\cos^2 A - 1$

  $= 1 - 2\sin^2 A$
- $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

Products as sums and differences

- $\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$
- $\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$
- $\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$
- $\cos A \sin B = \frac{1}{2}(\sin(A + B) - \sin(A - B))$

Pythagorean identities

- $\cos^2 A + \sin^2 A = 1$
- $\tan^2 A + 1 = \sec^2 A$
- $\cot^2 A + 1 = \mathrm{cosec}^2 A$

Reciprocal trigonometric functions

- $\sec A = \dfrac{1}{\cos A}$, $\cos A \neq 0$
- $\operatorname{cosec} A = \dfrac{1}{\sin A}$, $\sin A \neq 0$
- $\cot A = \dfrac{\cos A}{\sin A}$, $\sin A \neq 0$

Matrices

Addition of matrices (See Matrix)

If ***A*** and ***B*** are matrices with the same dimensions and the entries of ***A*** are a_{ij} and the entries of ***B*** are b_{ij} then the entries of ***A* + *B*** are $a_{ij} + b_{ij}$

For example if ***A*** = $\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$ and ***B*** = $\begin{bmatrix} 5 & 1 \\ 2 & 1 \\ 1 & 6 \end{bmatrix}$ then

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \\ 2 & 10 \end{bmatrix}$$

Determinant of a 2 × 2 matrix (See Matrix)

If ***A*** = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinant of ***A*** denoted as det ***A*** = $ad - bc$.

If det ***A*** ≠ 0,

- the matrix ***A*** has an **inverse**.
- the simultaneous linear equations $ax + by = e$ and $cx + dy = f$ have a unique solution.
- The linear transformation of the plane, defined by ***A*** maps the unit square $O(0, 0)$, $B(0,1)$, $C(1, 1)$, $D(1, 0)$ to a parallelogram $OB'C'D'$ of area |det ***A*** |.
- The sign of the determinant determines the orientation of the image of a figure under the transformation defined by the matrix.

Dimension (or size) (See Matrix)

Two matrices are said to have the same **dimensions** (or **size**) **i**f they have the same number of rows and columns.

- For example, the matrices

$$\begin{bmatrix} 1 & 8 & 0 \\ 2 & 5 & 7 \end{bmatrix} \text{and} \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

have the same dimensions. They are both 2×3 matrices.

- An $m \times n$ matrix has m rows and n columns.

Entries (Elements) of a matrix

The symbol a_{ij} represents the (i, j) entry which occurs in the i^{th} row and the j^{th} column. For example a general 3×2 matrix is:

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ and a_{32} is the entry in the third row and the second column.

Leading diagonal

The leading diagonal of a square matrix is the diagonal which runs from the top left corner to the bottom right corner.

Linear transformation defined by a 2 × 2 matrix

The matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

defines a transformation $T(x, y) = (ax + by, cx + dy)$.

Linear transformations in 2 dimensions

A linear transformation in the plane is a mapping of the form

$T(x, y) = (ax + by, cx + dy)$.

A transformation T is linear if and only if

$T(\alpha(x_1, y_1) + \beta(x_2, y_2)) = \alpha T((x_1, y_1)) + \beta T(x_2, y_2))$.

Linear transformations include:

- rotations around the origin
- reflections in lines through the origin
- dilations.

Translations are not linear transformations.

Matrix (matrices)

A **matrix** is a rectangular array of elements or entries displayed in rows and columns.

For example,

A = $\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$ and ***B*** = $\begin{bmatrix} 1 & 8 & 0 \\ 2 & 5 & 7 \end{bmatrix}$ are both matrices.

Matrix ***A*** is said to be a 3 × 2 matrix (three rows and two columns) while ***B*** is said to be a 2 × 3 matrix (two rows and three columns).

A **square matrix** has the same number of rows and columns.

A **column matrix** (or vector) has only one column.

A **row matrix** (or vector) has only one row.

Matrix algebra of 2 × 2 matrices

If ***A, B*** and ***C*** are 2 × 2 matrices, ***I*** the 2 × 2 (multiplicative) identity matrix and ***O*** the 2 × 2 zero matrix then:

A + ***B*** = ***B*** + ***A***	(commutative law for addition)
(***A*** + ***B***) + ***C*** = ***A*** + (***B*** + ***C***)	(associative law for addition)
A + ***O*** = ***A***	(additive identity)
A + (–***A***) = ***O***	(additive inverse)
(***AB***)***C*** = ***A***(***BC***)	(associative law for multiplication)
AI = ***A*** = ***IA***	(multiplicative identity)
A(***B*** + ***C***) = ***AB*** + ***AC***	(left distributive law)
(***B*** + ***C***)***A*** = ***BA*** + ***CA***	(right distributive law)

Matrix multiplication

Matrix multiplication is the process of multiplying a matrix by another matrix. The product ***AB*** of two matrices ***A*** and ***B*** with **dimensions** $m \times n$ and $p \times q$ is defined if $n = p$. If it is defined, the product ***AB*** is an $m \times q$ matrix and it is computed as shown in the following example.

$$\begin{bmatrix} 1 & 8 & 0 \\ 2 & 5 & 7 \end{bmatrix}\begin{bmatrix} 6 & 10 \\ 11 & 3 \\ 12 & 4 \end{bmatrix} = \begin{bmatrix} 94 & 34 \\ 151 & 63 \end{bmatrix}$$

The entries are computed as shown

$1 \times 6 + 8 \times 11 + 0 \times 12 = 94$

$1 \times 10 + 8 \times 3 + 0 \times 4 = 34$

$2 \times 6 + 5 \times 11 + 7 \times 12 = 151$

$2 \times 10 + 5 \times 3 + 7 \times 4 = 63$

The entry in row i and column j of the product ***AB*** is computed by 'multiplying' row i of ***A*** by column j of **B** as shown.

If ***A*** = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ and ***B*** = $\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ then

$$AB = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} & a_{11}b_{13}+a_{12}b_{23} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} & a_{21}b_{13}+a_{22}b_{23} \\ a_{31}b_{11}+a_{32}b_{21} & a_{31}b_{12}+a_{32}b_{22} & a_{31}b_{13}+a_{32}b_{23} \end{bmatrix}$$

(Multiplicative) identity matrix

A **(multiplicative) identity matrix** is a square matrix in which all the elements in the leading diagonal are 1s and the remaining elements are 0s. Identity matrices are designated by the letter ***I***.

For example,

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are both identity matrices.

There is an identity matrix for each order of square matrix. When clarity is needed, the order is written with a subscript: I_n

Multiplicative inverse of a square matrix

The inverse of a square matrix ***A*** is written as A^{-1} and has the property that

$$AA^{-1} = A^{-1}A = I$$

Not all square matrices have an inverse. A matrix that has an inverse is said to be **invertible**.

multiplicative inverse of a 2 × 2 matrix

The **inverse** of the matrix ***A*** = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, when det ***A*** ≠ 0.

Scalar multiplication (matrices)

Scalar multiplication is the process of multiplying a matrix by a scalar (number).

For example, forming the product:

$$10\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 0 & 30 \\ 10 & 40 \end{bmatrix}$$

is an example of the process of scalar multiplication.

In general for the matrix ***A*** with entries a_{ij} the entries of k***A*** are ka_{ij}.

Singular matrix

A matrix is singular if det ***A*** = 0. A singular matrix does not have a multiplicative inverse.

Zero matrix

A zero matrix is a matrix if all of its entries are zero. For example:

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are zero matrices.

There is a zero matrix for each **size** of matrix. When clarity is needed we write $\mathbf{O}_{n \times m}$ for the $n \times m$ zero matrix.

Real and Complex Numbers

Complex numbers

Complex arithmetic

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

- $z_1 + z_2 = x_1 + x_2 + (y_1 + y_2)\,i$
- $z_1 - z_2 = x_1 - x_2 + (y_1 - y_2)\,i$
- $z_1 \times z_2 = x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1)i$
- $z_1 \times (0 + 0i) = 0$ Note: $0 + 0i$ is usually written as 0
- $z_1 \times (1 + 0i) = z_1$ Note: $1 + 0i$ is usually written as 1

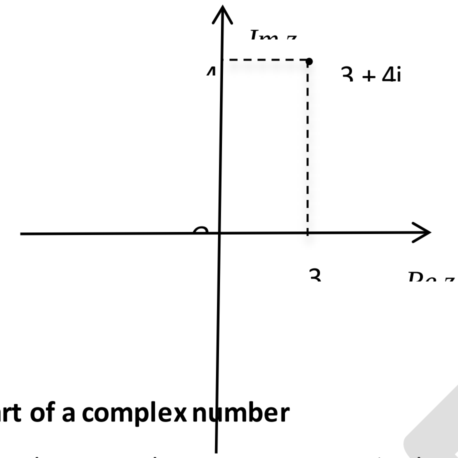
Complex conjugate

For any complex number $z = x + iy$, its **conjugate** is $\bar{z} = x - iy$. The following properties hold

- $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$
- $\overline{z_1 / z_2} = \bar{z}_1 \, / \, \bar{z}_2$
- $z\bar{z} = |z|^2$
- $z + \bar{z}$ is real

Complex plane (Argand plane)

The **complex plane** is a geometric representation of the complex numbers established by the **real axis** and the orthogonal **imaginary axis**. The complex plane is sometimes called the Argand plane.



Imaginary part of a complex number

A complex number z may be written as $x + yi$, where x and y are real, and then y is the imaginary part of z. It is denoted by $Im(z)$.

Integers

The **integers** are the numbers $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$.

Modulus (Absolute value) of a complex number

If z is a complex number and $z = x + iy$ then the modulus of z is the distance of z from the origin in the Argand plane. The modulus of z denoted by $|z| = \sqrt{x^2 + y^2}$.

Prime numbers

A prime number is a positive integer greater than 1 that has no positive integer factor other 1 and itself. The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23, \cdots$.

Rational numbers

A real number is **rational** if it can be expressed as a quotient of two integers. Otherwise it is called irrational.

Irrational numbers can be approximated as closely as desired by rational numbers, and most electronic calculators use a rational approximation when performing calculations involving an irrational number.

Real numbers

The numbers generally used in mathematics, in scientific work and in everyday life are the **real numbers**. They can be pictured as points on a number line, with the integers evenly spaced along the line, and a real number a to the right of a real number b if $a > b$.

A real number is either rational or irrational. The set of real numbers consists of the set of all rational and irrational numbers.

Every real number has a decimal expansion. Rational numbers are the ones whose decimal expansions are either terminating or eventually recurring.

Real part of a complex number

A complex number z may be written as $x + yi$, where x and y are real, and then x is the real part of z. It is denoted by $Re\ (z)$.

Whole numbers

A **whole number** is a non–negative integer, that is, one of the numbers $0, 1, 2, 3, \cdots$,

Graph Theory

Adjacent (graph)

Vertices are **adjacent** if they are connected by an edge.

Bipartite Graph

A **bipartite graph** is a graph whose set of vertices can be split into two disjoint subsets *A* and *B* in such a way that each edge of the graph joins a vertex in *A* and a vertex in *B*.

Circuit and Cycle

A **trail** whose end vertices coincide (a closed trail) is called a **circuit**.

If a walk with vertices x_i, in the sequence $x_{1,} x_2, \ldots, x_n$ with $n \geq 3$ and the vertices x_i with $1 < i < n$ distinct from each other and from x_1 and $x_1 = x_n$ then it is called a **cycle.**

Complete graph

A **complete graph** is a **simple graph** in which every vertex is joined to every other vertex by an edge.

The complete graph with n vertices is denoted by K_n

Complete Bipartite Graph

A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by exactly one edge. The complete bipartite graph with *r* vertices in A and *s* vertices in B is denoted by $K_{r,s}$

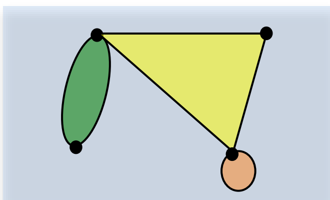
Connected graphs and disconnected graphs

A graph is **connected** if there is a path between each pair of vertices and **disconnected** otherwise.

Degree (of a vertex). The degree of a vertex v_i in a graph is the number of edges incident with v_i, with each loop counted twice.

Face

The faces of a planar graph are the regions bounded by the edges including the outer infinitely large region. The planar graph shown has 4 faces.



Euler circuit

An **Euler circuit** is a **circuit** that includes each **edge** in a **graph**.

Euler's formula

In a connected **planar graph**

$$v - e + f = 2$$

where v is the number of vertices, e the number of edges and f is the number of faces including the exterior face. This formula relating the numbers of vertices, edges and faces is known as **Euler's formula**.

Euler trail

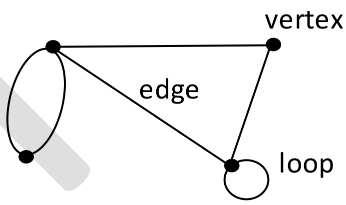
A trail of a graph containing all edges of the graph is an **Euler trail.** A connected simple graph has an Euler trail if and only if each of the vertices has even degree.

Fleury's algorithm

An algorithm to find an Euler trail.

Graph

A **graph** is a diagram that consists of a set of points, called **vertices** that are joined by a set of lines called **edges**. Each edge joins two vertices. A **loop** is an edge in a graph that joins a **vertex** in a **graph** to itself.



Hamiltonian cycle

A **Hamiltonian cycle** of a graph is a **cycle** that includes each **vertex** in that **graph**.

Hamiltonian path

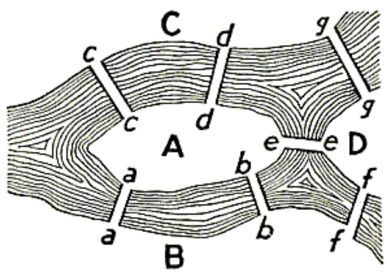
A **Hamilton path** of a graph is a **path** containing all vertices of the graph.

Isomorphism of graphs

Let G be a (labelled) graph. We say that a graph H is isomorphic to G if H can be labelled with the same "letters" used to label G in such a way that the pairs of vertices in the two graphs corresponding to the same pair of labels are joined by the same number of edges (same number of edges in both graphs).

Königsberg bridge problem

The Königsberg bridge problem asks if the seven bridges of the city of Königsberg can all be traversed in a single trip without crossing the same bridge twice, with the additional requirement that the trip ends in the same place it began.



Labelled graph

A **labelled graph** is a graph with each vertex labelled differently (but arbitrarily), usually with letters of the alphabet.

Paths, Walks and Trails

A **walk** in a graph is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n$, of vertices v_i and edges e_i. A walk is often described by simply listing the sequence of edges $e_1, e_2, \ldots, e_{n-1}$, or when there is no confusion, by simply listing the sequence of vertices $v_1, v_2, \ldots, v_n$.

A **path** in a graph is a walk in that graph with all of its vertices distinct.

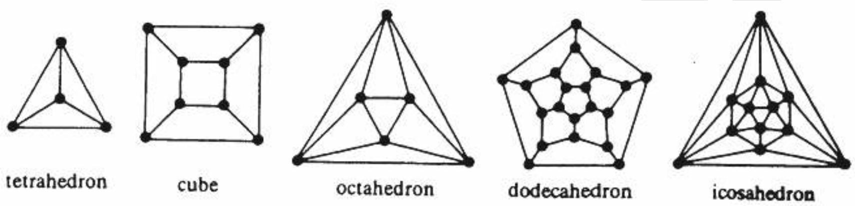
A **trail** in a graph is a walk in that graph with all of its edges are distinct.

Planar graph

A planar graph is a graph that can be drawn in 2 dimensions in such a way that no two edges cross.

Platonic graphs

The five regular planar graphs shown here are known as the Platonic graphs.

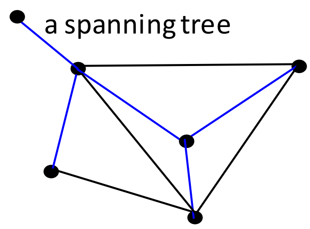


Regular graph

A graph is regular if its vertices all have the same degree.

Simple graph

A **simple graph** has no **loops** or **multiple edges.**



Spanning tree

A **spanning tree** is **subgraph** of a **connected graph** that connects all vertices and is also a **tree.**

Subgraph

A **subgraph** of a graph *G* is a graph all of whose vertices are vertices of *G* and all of whose edges are edges of *G.*

Trail (see Paths, Walks and Trail)

Tree

A **tree** is a connected graph with no circuits.

Walk (see Paths, Walks and Trail)

Unit 3

Vectors in three-dimensions

(See Vectors in Unit 2)

Addition of vectors

In component form if **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k** then

a + **b** = $(a_1 + b_1)$**i** + $(a_2 + b_2)$**j** + $(a_3 + b_3)$**k**

Vector equation of a straight line

Let **a** be the position vector of a point on a line l, and **u** any vector with direction along the line. The line consists of all points *P* whose position vector **p** is given by

p = **a** + t**u** for some real number t.

(Given the position vectors of two points on the plane **a** and **b** the equation can be written as

p = **a** + t(**b** – **a**) for some real number t.)

Vector equation of a plane

Let **a** be a position vector of a point *A* in the plane, and **n** a normal vector to the plane . Then the plane consists of all points *P* whose position vector **p** satisfies

(**p** – **a**).**n** = **0**. This equation may also be written as **p.n** =**a.n,** a constant.

(If the the normal vector **n** is the vector (l, m, n) in ordered triple notation and the scalar product

a.n = k, this gives the Cartesian equation $lx + my + nz = k$ for the plane)

Vector function

In this course a vector function is one that depends on a single real number parameter t, often representing time, producing a vector ***r***(t) as the result. In terms of the standard unit vectors **i**, **j**, **k** of three dimensional space, the vector-valued functions of this specific type are given by expressions such as

r$(t) = f(t)$**i** + $g(t)$**j** + $h(t)$**k**

where *f*, *g* and *h* are real valued functions giving coordinates.

Scalar product

If **a** = (a_1, a_2, a_3) and **b** = (b_1, b_2, b_3) then the scalar product **a**.**b** is the real number

$a_1b_1 + a_2b_2 + a_3b_3$.

When expressed in **i**, **j**, **k** notation, **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k** then

a.b = $a_1b_1 + a_2b_2 + a_3b_3$

Vector product (Cross product)

When expressed in **i**, **j**, **k** notation, **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k** then

a × **b** = $(a_2b_3 - a_3b_2)$**i** + $(a_3b_1 - a_1b_3)$**j** + $(a_1b_2 - a_2b_1)$**k**

The cross product has the following geometric interpretation. Let **a** and **b** be two non- parallel vectors then |**a** × **b**| is the area of the parallelogram defined by **a** and **b** and

a × **b** is normal to this parallelogram.

(The cross product of two parallel vectors is the zero vector.)

Matrices and Systems of Equations

Augmented matrix

For a given set of m linear equations in n variables:

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

. . . .

$\vdots$ $a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_n$

the augmented matrix is $\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$

Row-echelon form

A matrix is in row echelon form if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes, and
- The leading coefficient (the first nonzero number from the left) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

Real and Complex Numbers

Argument (abbreviated arg)

If a complex number is represented by a point P in the complex plane then the argument of z, denoted arg z, is the angle θ that OP makes with the positive real axis O_x, with the angle measured anticlockwise from O_x. The **principal value** of the argument is the one in the interval $(-\pi, \pi]$.

Complex arithmetic (see Complex arithmetic Unit 2)

Complex conjugate (see Complex conjugate Unit 2)

De Moivre's Theorem

For all integers n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Modulus of a complex number (Modulus of a complex number Unit 2)

Polar form of a complex number

For a complex number z, let $\theta = \arg z$. Then $z = r(\cos\theta + i\sin\theta)$ is the polar form of z.

Root of unity (nth root of unity)

A complex number z such that $z^n = 1$

The n^{th} roots of unity are:

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \qquad \text{where } k = 0, 1, 2, \ldots, n-1.$$

The points in the complex plane representing roots of unity lie on the unit circle.

The cube roots of unity are

$$z_1 = 1,\ z_2 = \frac{1}{2}(-1 + i\sqrt{3}),\ z_3 = \frac{1}{2}(-1 - i\sqrt{3}).\ \text{Note } z_3 = \bar{z}_2 \text{ and } z_3 = \frac{1}{z_2} \text{ and } z_2 z_3 = 1.$$

Calculus

Inverse Trigonometric functions

The inverse sine function, $y = \sin^{-1} x$

If the domain for the sine function is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ a one to one function is formed and so an inverse function exists.

The inverse of this restricted sine function is denoted by $\sin^{-1}$ and is defined by:

$$\sin^{-1}: [-1, 1] \to R,\ \sin^{-1} x = y \text{ where } \sin y = x,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\sin^{-1}$ is also denoted by $\arcsin$.

The inverse cosine function, $y = \cos^{-1} x$

If the domain of the cosine function is restricted to $[0, \pi]$ a one to one function is formed and so the inverse function exists.

$\cos^{-1} x$, the inverse function of this restricted cosine function, is defined as follows:

$$\cos^{-1}: [-1, 1] \to R,\ \cos^{-1} x = y \text{ where } \cos y = x,\ y \in [0, \pi]$$

$\cos^{-1}$ is also denoted by arccos.

The inverse tangent function, $y = \tan^{-1} x$

If the domain of the tangent function is restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$ a one to one function is formed and so the inverse function exists.

$$\tan^{-1}: R \to R,\ \tan^{-1} x = y \text{ where } \tan y = x,\ y \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$\tan^{-1}$ is also denoted by arctan.

Rational function

A rational function is a function such that $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials. Usually $g(x)$ and $h(x)$ are chosen so as to have no common factor of degree greater than or equal to 1, and the domain of f is usually taken to be
$R \backslash \{x: h(x) = 0\}$.

Volume of a solid of revolution

Let $y = f(x)$ be the graph of a function f, a continuous function on $[a, b]$, and such that $f(x) \geq 0$ for all $x \in [a, b]$. If the region bounded by the x axis, $y = f(x)$ and the lines $x = a$ and $x = b$ is rotated around the x axis the volume of the resulting solid is $\pi \int_a^b [f(x)]^2\, dx$

Unit 4

Further Calculus and Applications of Calculus

Linear momentum

The linear momentum $\mathbf{p}$ of a particle is the vector quantity $\mathbf{p} = m\mathbf{v}$ where m is the mass and $\mathbf{v}$ is the velocity.

Logistic equation

The logistic equation has applications in a range of fields, including biology, biomathematics, economics, chemistry, mathematical psychology, probability, and statistics.

One form of this differential equation is:

$$\frac{dy}{dt} = ay - by^2 \qquad \text{(where } a > 0 \text{ and } b > 0\text{)}$$

The general solution of this is

$y = \dfrac{a}{b + Ce^{-at}}$, where C is an arbitrary constant.

Separation of variables

Differential equations of the form $\dfrac{dy}{dx} = g(x)h(y)$ can be rearranged as long as $h(y) \neq 0$ to obtain

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x).$$

Slope field

Slope field (**direction or gradient field**) is a graphical representation of the solutions of a linear first-order differential equation in which the derivative at a given point is represented by a line segment of the corresponding slope

Statistical Inference for Continuous data

Continuous random variable

A random variable X is called continuous if its set of possible values consists of intervals, and the chance that it takes any point value is zero (in symbols, if $P(X = x) = 0$ for every real number x). A random variable is continuous if and only if its cumulative probability distribution function can be expressed as an integral of a function.

Probability density function

The probability density function (pdf) of a continuous random variable is the function that when integrated over an interval gives the probability that the continuous random variable having that pdf, lies in that interval. The **probability density function** is therefore the derivative of the (cumulative probability) distribution function.

Continuous data

Continuous data are observations or measurements of a continuous random variable.

Matched data pairs

Matched data pairs are sometimes called matched samples. These can arise in the following situations:

a. Two samples in which the members are clearly paired, or are matched explicitly by the researcher. For example, IQ measurements on pairs of identical twins.

b. Those samples in which the same attribute, or variable, is measured twice on each subject, under different circumstances. Commonly called repeated measures. Examples include the times of a group of athletes for 1500m before and after a week of special training; or the milk yields of cows before and after being fed a particular diet.

Often the difference in the value of the measurement of interest for each matched pair is calculated, for example, the difference between before and after measurements, and these figures then form a single sample for an appropriate statistical analysis.

Interpretation of patterns in matched pair situations

To investigate comparisons of measurements on pairs of subjects or in before- and after- situations, it is customary to study the set of differences. This can be done informally by studying appropriately-chosen graphs; more formally, binomial probabilities can be used to investigate whether, in the event of no actual difference, there are more differences of one sign than could be reasonably expected just due to chance. If differences are equally likely to be positive or negative, the number of differences out of n (the sample size) that are of one sign has a binomial distribution with $p = 0.5$, and the probability of obtaining at least the observed number of differences of one sign can be obtained. If this is small, there was little possibility of obtaining such data just due to chance, thereby providing evidence of a difference between the pairs.

Difference between paired means

Mean of the difference between matched pairs

Least squares line

Suppose that a sample of pairs of measurements of two variables (X, Y) is available, and we seek to estimate the average value of Y for a given value of $X = x$, using a linear function of X. The least squares line is the straight line that minimises the sum of squared distances between the sample values y of Y and the predicted values $a + bx$.

Permutations

Re-arrangements of a sequence of values. Random permutations of a set of data may be made by randomly assigning their labels using random numbers.

Estimate the probability of obtaining values of slopes by chance

After random re-arrangements of the *y*-values, and calculating the slopes of the corresponding least squares lines, obtain the proportion of these slopes that are at least as far from 0 as the slope of the observed least squares line for the original data. This proportion is an estimate of the probability of obtaining at least the slope of the observed least squares line if there is in fact no linear relationship between *x* and *y*. A small value of this probability therefore indicates that the observed data were unlikely if there is no linear relationship between *x* and *y*, and hence provides evidence that there is. The smaller the value of this probability, the greater the evidence against the assumption of no linear relationship between *x* and *y*.

Precision

Precision is a measure of how close an estimator is expected to be to the true value of the parameter it purports to estimate.

Independent and identically distributed observations

For independent observations, the value of any one observation has no effect on the chance of values for all the other observations. For identically distributed observations, the chances of the possible values of each observation are governed by the same probability distribution.

Random sample

A random sample is a set of data in which the value of each observation is governed by some chance mechanism that depends on the situation. The most common situation in which the term "random sample" is used refers to a set of independent and identically distributed observations.

Sample median For a sample of data of odd size, the middle value when the data are arranged in increasing order; for a sample of even size, the average of the two middle values.

Sample mean the arithmetic average of the sample values

Uniform distribution the distribution of a continuous random variable with constant probability density function 1/*a* over an interval of length *a.*