

# Saint Ignatius' College Riverview

## Mathematics Extension 2

### Trial HSC Examination 2001

- (a) If  $P = 2 - ti$  where  $t$  is real, find  $\overline{iP}$ .

(b) The complex number,  $u$ , is given by  $u = \frac{\sqrt{3}-i}{1+i}$   
Find (i) The modulus of  $u$  (ii) The exact value of  $\arg u$  (iii)  $u^6$  in the form  $a + bi$

(c) On an Argand diagram shade the region satisfied by both of the conditions:  
 $|z - 2| \geq 1$  and  $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}$ .

(d) If the complex number  $Q = \frac{z-1}{z-2i}$ , is purely imaginary (where  $z = x + iy$ ) determine the Cartesian equation for the locus of  $z$  and sketch this locus.
- (a) Find  $\int \operatorname{cosec} \theta \, d\theta$  using  $t = \tan \frac{\theta}{2}$

(b) Evaluate  $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dy}{1+\cos y}$

(c) Evaluate  $\int_0^{\frac{\pi}{4}} e^x \sin^2 x \, dx$

(d) Find  $\int \frac{t^2-t-21}{(t^2+4)(2t-1)} \, dt$
- (a) Consider the function  $f(x) = 9 - x^2$ . On three separate sets of axes, sketch the following, showing all important features.

(i)  $y = f(x)$  (ii)  $y = |f(x)|$  (iii)  $|y| = f(x)$

(b) Consider the function  $y = \sin(\cos^{-1} x)$

(i) Find the domain and range of the function.

(ii) Sketch this function showing the important features.

(c) Consider the function  $f$  and  $g$  defined by:  
 $f(x) = \frac{x+1}{x-2}$  for  $x \neq 2$  and  $g(x) = [f(x)]^2$

(i) Sketch the hyperbola  $y = f(x)$ , clearly labelling the horizontal and vertical asymptotes and the points of intersection with the  $x$  and  $y$  axes.

(ii) Sketch the curve  $y = g(x)$  on a separate diagram showing all important features including any turning points.

(iii) On a separate diagram sketch the curve given by  $y = g(-x)$
- (a) (i) On the same number plane sketch the graphs of  $y = |x| - 3$  and  $y = 5 + 4x - x^2$

(ii) Hence, or otherwise, solve  $\frac{|x|-3}{5+4x-x^2} > 0$

(b) Given the polynomial  $Q(x)$ , where  $Q(x) = kx^{k+1} - (k+1)x^k + 1$  ( $k \neq 0$ ) prove that  $Q(x)$  is divisible by  $(x-1)^2$ .

(c) The equation  $x^3 + 2x - 1 = 0$  has roots  $p, q$  and  $r$ . Find

(i) the value of  $p^2 + q^2 + r^2$

(ii) the equation with roots  $-p, -q$  and  $-r$

(d)  $P(x)$  is a polynomial with the following form:

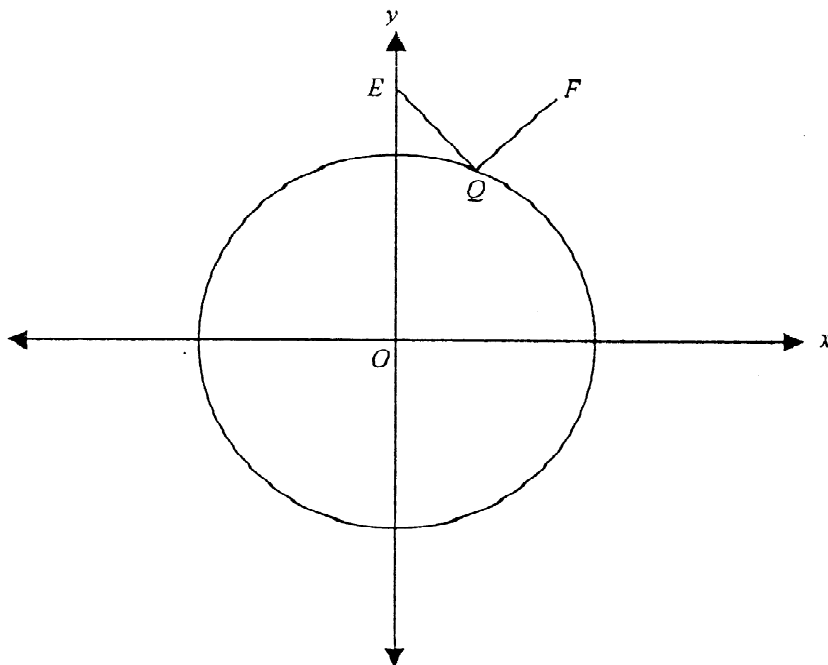
$$P(x) = Kx^3 + Mx^2 + Lx + N \text{ where } K, M, L \text{ and } N \text{ are real.}$$

$P(x)$  has roots of 5 and  $i$  and when divided by  $(x - 2)$  the remainder is 3. Find  $P(x)$ .

5. (a) (i) Show, using differentiation, that the equation of the normal to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at the point  $P(a \sec \alpha, b \tan \alpha)$  is  $ax \tan \alpha + by \sec \alpha = (a^2 + b^2) \sec \alpha \tan \alpha$

(ii) The vertical line through  $P$  meets an asymptote of the above hyperbola at  $M$ . The normal at  $P$  meets the  $x$  axis at  $K$ . Show that  $KM$  is at right angles to the asymptote.

(b) The following diagram shows a circle with centre at the origin  $O$ . The point  $E(0, a)$  is fixed where  $a > 3$ .  $Q$  lies on the circle such that the angle  $EQF$  is a right angle and  $EQ = QF$ .



(i) Copy the diagram.

(ii) Show by substitution that  $Q(3 \cos \alpha, 3 \sin \alpha)$  satisfies  $x^2 + y^2 = 9$

(iii) Prove, using congruent triangles or otherwise, that  $F$  has coordinates  $(3 \cos \alpha + a - 3 \sin \alpha, 3 \cos \alpha + 3 \sin \alpha)$

(iv) Find the locus of  $F$  as  $Q$  moves on the circle.

(v) Prove that the locus of the circle is independent of the value of  $a$ .

6. (a) The base of a solid is the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  where  $b > a$ . Sections perpendicular to the  $y$ -axis are squares with one side in the base of the solid. Show that the volume of the solid is  $\frac{16a^2b}{3}$  cubic units.

(b) The curve  $y = \sin x$  is revolved about the straight line  $y = 1$ . Use a slicing technique to find the volume of the solid of revolution formed by the portion of the curve from  $x = 0$  to  $x = \frac{\pi}{2}$ .

(c) The area enclosed by  $y = (x-2)^2$  and the straight line  $y = 4$  is rotated about the  $y$ -axis. Using the method of cylindrical shells, find the volume of the solid formed.

7. (a) A certain particle of unit mass moving through air experiences air resistance proportional to the square of its speed,  $v$  metres per second.

(i) Explain why the equations of motion with upwards taken as positive are:

$$\ddot{x} = -g - kv^2, \text{ when moving upwards and}$$

$$\ddot{x} = -g + kv^2, \text{ when moving downwards,}$$

where  $g$  is the acceleration due to gravity and  $k$  is a positive constant.

(ii) Suppose that the particle is fired vertically upwards from the ground with an initial speed of  $u$  metres per second.

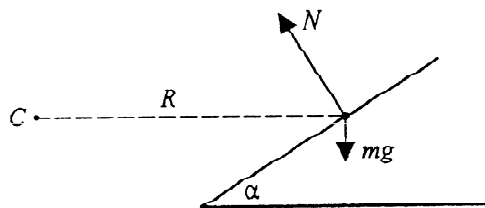
(α) Show that the maximum height,  $H$  metres, reached by the particle is:

$$H = \frac{1}{2k} \ln\left(1 + \frac{ku^2}{g}\right)$$

(β) Show that the time,  $T$  seconds, taken to reach maximum height is:

$$T = \frac{1}{\sqrt{gk}} \tan^{-1}\left(\frac{u\sqrt{k}}{\sqrt{g}}\right)$$

(b) (i) A particle of mass  $m$  travels with constant velocity  $v$  in a horizontal circle of radius  $R$ , centre  $C$ , around a track banked at an angle  $\alpha$  to the horizontal, as shown in the diagram.



Show that if there is no tendency for the particle to slip sideways then  $v = \sqrt{Rg \tan \alpha}$ .

(ii) A particle travels in a horizontal circle of radius 1 metre around the lower half of the track where the angle of banking is given by  $\tan^{-1}(\frac{5}{18})$ . Another particle travels in a horizontal circle of radius 1.2 metres around the upper half of the track where the angle of banking is given by  $\tan^{-1}(\frac{16}{27})$ . Each particle travels with constant velocity so that it has no tendency to slip sideways. The particles are initially observed to be alongside each other. Taking  $g = 10$  metres per second squared, find the time that elapses before the particles are next observed to be alongside each other.

8. (a) Prove that:

(i)  $p^2 + q^2 \geq 2pq$

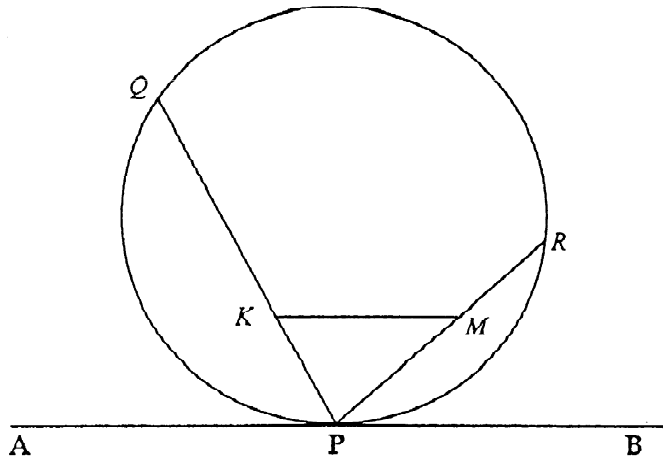
(ii)  $k^4 + l^4 + m^4 + n^4 \geq 4klmn$ , if  $k, l, m$  and  $n$  are positive.

(b) Given  $z = \cos \alpha + i \sin \alpha$ , where  $\sin \alpha \neq 0$ :

(i) Prove that  $\frac{1}{1-z \cos \alpha} = 1 + i \cot \alpha$ .

(ii) Hence, by considering  $\sum_{k=0}^{\infty} (z \cos \alpha)^k$ , deduce the sum of the infinite series  $\sin \alpha \cos \alpha + \sin 2\alpha \cos^2 \alpha + \cdots + \sin k\alpha \cos^k \alpha + \cdots$ .

(c)



$AB$  is a tangent to the circle at  $P$ .  $M$  and  $K$  move on  $PR$  and  $PQ$  respectively so that  $KM$  is parallel to  $AB$ . Prove that the point of intersection of the perpendicular bisectors of  $QK$  and  $RM$  moves on a straight line.