SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2007

MATHEMATICS EXTENSION 2

12:30pm – 3:35pm
Tuesday 28th August 2007
Directions to Students

• Reading Time : 5 minutes	• Total Marks **120**
• Working Time : 3 hours	
• Write using blue or black pen. (sketches in pencil).	• Attempt Question 1 – 8
• Board approved calculators may be used	• All questions are of equal value
• A table of standard integrals is provided at the back of this paper.	
• All necessary working should be shown in every question.	
• **Answer each question in the booklets provided and clearly label your name and teacher's name.**	

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ 1

(ii) Hence find the indefinite integral $\int \sin 5x \cos 3x \, dx$ 2

(b) Evaluate $\displaystyle\int_0^5 \frac{t\,dt}{\sqrt{t+4}}$ 2

(c) Evaluate $\displaystyle\int_{-1}^1 3^x\,dx$ correct to three significant figures. 2

(d) Evaluate $\displaystyle\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{1-9x^2}}$ 2

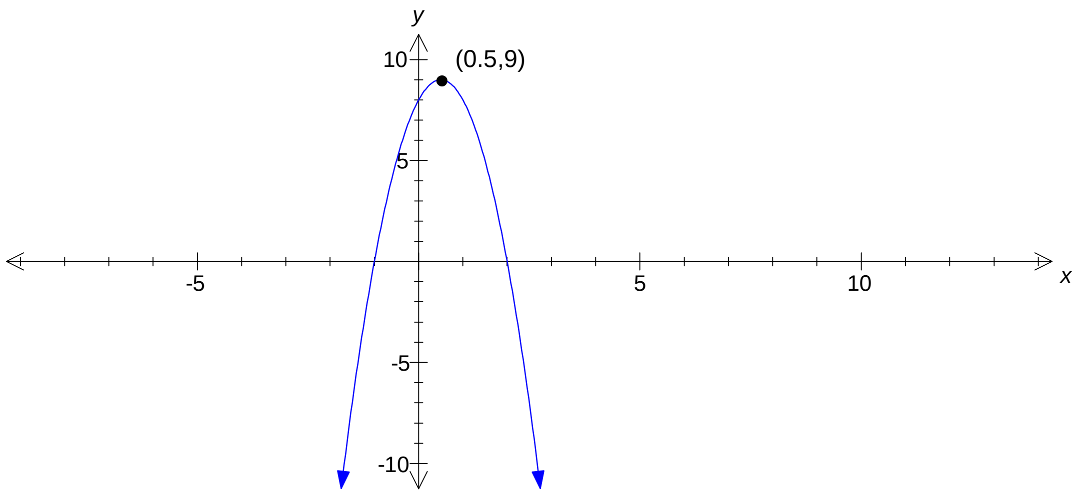
(e) Find $\displaystyle\int \frac{1}{1+\sin x}dx$, using the substitution $t=\tan\dfrac{x}{2}$ 3

(f) Use integration by parts to find $\displaystyle\int \frac{\cos^{-1}x}{\sqrt{1+x}}dx$ 3

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove that $f(x)=\dfrac{x^3}{\sin x}$ is an even function. 2

(b) A sketch of $f(x)=-4(x+1)(x-2)$ is shown below.



With the aid of the above diagram, and without the use of calculus, draw a separate half page sketch for each of the following.

(i) $y=|f(x)|$ 1

(ii) $y=f(2x)$ 1

(iii) $y=f(-x)$ 1

(iv) $y=\dfrac{1}{f(x)}$ 2

(v) $y=\sqrt{f(x)}$ 2

(vi) $y=\log_e f(x)$ 2

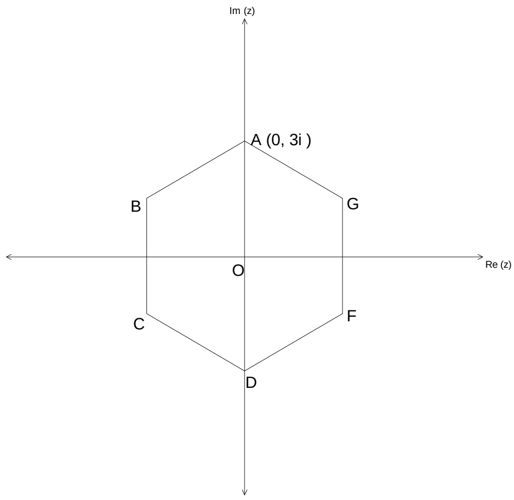
(c) Using calculus, show that $e^{-x}+x-1\geq 0$ for real x. 4

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Express the following in the form $(x+iy)$, where x and y are real: 2

$$\frac{i^2-1}{i}+\frac{1}{1+i}$$

(b)



The Argand diagram above, shows a regular hexagon with vertex A at the point $(0,3i)$. O is the centre of the hexagon.

(i) Copy the diagram into your writing booklet.

(ii) On your diagram show the region within the hexagon in which both the inequalities $|z|\leq 2$ and $-\frac{\pi}{6}\leq \arg z\leq \frac{\pi}{6}$ are satisfied. 2

(iii) Find in the form $|z-z_1|=R$, the equation of the circle through the points O, B and F. 1

(iv) Find the complex numbers, in modulus argument form, represented by the points B and C. 2

(v) The hexagon is rotated anticlockwise about the origin through an angle of $\frac{\pi}{4}$. Express in the form $r(\cos\theta+i\sin\theta)$, where θ is the principal argument, the complex numbers represented by the new positions of B and C. 3

Question 3 continues on page 5

Question 3 (continued)

(c) If $1 - 2i$ is a root of the equation $z^2 - (3+i)z + k = 0$,

(i) explain why the conjugate $1 + 2i$ cannot be a root to the equation 1

(ii) show that the other root is $2 + 3i$ 1

(iii) find the value of k 1

(iv) hence, or otherwise, find the two square roots of $-24 + 10i$. 2

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A polynomial $p(x)=x^n+ax^2-2$ has a factor of $(x-1)$ and leaves a remainder of -6 on division by $(x+2)$.

Find:

(i) the value of a 1

(ii) the value of n 1

(iii) the zeros of $p(x)$. 2

(b) Find the values of a and b so that $p(x)=2x^3-(2a+1)x^2+(2+b)x-1$ has a double root at $x=1$. 4

(c) If l, m, n are the roots of the equation $x^3-2x+5=0$,

(i) find the cubic equation whose roots are $2l, 2m, 2n$. 2

(ii) find the value of $l^3+m^3+n^3$. 2

(d) Find all the values of k for which the polynomial equation $3x^4-4x^3+k=0$ has no real roots. 3

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) An ellipse has the equation $\frac{x^2}{25}+\frac{y^2}{16}=1$. O is the centre of the ellipse and S and S' are the foci.

(i) Find

(α) the eccentricity 1

(β) the co-ordinates of the foci 1

(γ) the equations of the directrices. 1

(ii) Make a third of a page sketch of the ellipse showing the features found in part (i) 2

(iii) If $P(x_0, y_0)$ is a point on the ellipse show that $(PS+PS')$ is constant. You may mark point P in quadrant one of the above mentioned diagram. 2

(iv) Show that the equation of the tangent at $P(x_0, y_0)$ is $\frac{xx_0}{25}+\frac{yy_0}{16}=1$. 2

(v) The tangent at P meets the nearer focus at R. If S is the nearer focus to P,

(α) write down the co-ordinates of R. 1

(β) find expressions for the gradients of PR and SR in terms of x_0 and y_0. 2

(γ) show that the angle PSR is a right angle. 1

(b) A hyperbola has its centre at the origin and asymptotes $y=\pm\frac{2}{3}x$. Find its equation. 2

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The region bounded by the curve $y = x(2 - x)$ and the x–axis is rotated about the y-axis. Find the volume of the solid of revolution by taking slices perpendicular to the y-axis. 4

(b) The region bounded by the curve $y = \ln x$, the line $y = 1$ and the co-ordinate axes is rotated about the x-axis.

(i) By dividing the resulting solid into cylindrical shells, show that each shell has an approximate volume : $\delta v = 2\pi y e^{y}\,\delta y$ where δy is the thickness of the shell. 2

(ii) Hence calculate the volume of the solid. 2

(c) The base of a particular solid is the circle $x^2 + y^2 = 8$. Find the volume of the solid if every cross section to the x-axis is an isosceles – right angled triangle with the hypotenuse in the base of the solid. 4

(d) Show that the straight line $lx + my + n = 0$ is a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ if $a^2l^2 - b^2m^2 = n^2$. 3

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Mr Kirkpatrick's mathematics class brought him a ride in a Gondola at Queenstown in New Zealand, during a recent trip. When Mr Kirkpatrick's hands were H metres above the Earth's surface, he dropped overboard his packet of beer nuts of mass m kg. The packet of beer nuts encounters air resistance proportional to its velocity v (which is in metres per second), that is the resistive force is equal to mkv.
Taking Mr Kirkpatrick's hands as the origin and downwards displacement as positive:

(i) Write down an equation of motion representing the passage of the packet of beer nuts. 2

(ii) Find the terminal velocity , w, of the packet of beer nuts. 1

(iii) Show that the equation of motion in part (i) can be written as $\ddot{x} = k(w - v)$. 1

(iv) Show that the displacement, x metres, of the packet of beer nuts from Mr Kirkpatrick's hands is given by: $x = -\frac{v}{k} - \frac{w}{k}\ln\left(\frac{w-v}{w}\right)$. 4

(v) If the packet reaches the Earth's surface with a velocity of u metres per second, show that $\ln\left(1 - \frac{u}{w}\right) + \frac{u}{w} + \frac{kH}{w} = 0$. 1

(vi) Consider the moment when the packet of beer nuts has reached 75% of its terminal velocity.
Find:

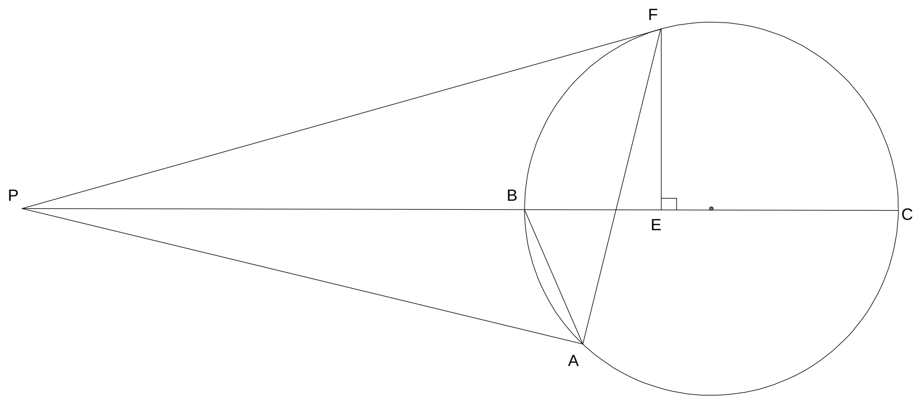
(α) the time, t seconds, for this moment to be reached. 3

(β) the distance fallen at this moment. 3

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Draw a neat half page sketch of the graph for $y^2 = x^2\left(4 - x^2\right)$. 3

(b)



In the above diagram AB and BC are chords of a circle, and F is on the arc ABC such that arc AF is equal to arc FC. E is the foot of the perpendicular from F to the chord BC. CB is extended to P so that $PE = EC$. (Note that B is inside the triangle APF)

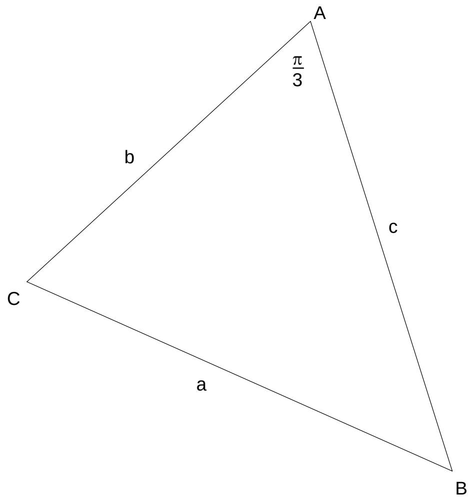
(i) Show that the triangle APF is isosceles. 3

(ii) Show that $AB + BE = EC$. 4

Question 8 continues on page 11

Question 8 (continued)

(c)



A triangle ABC has sides of varying length a, b and c with a fixed interior angle of $BAC =$ as shown in the above diagram.
Use the cosine rule to show that:

(i) $a^2 \geq bc$, and hence, 3

(ii) the area of triangle $ABC \leq \frac{a^2\sqrt{3}}{4}$ $ABC \leq \frac{a^2\sqrt{3}}{4}$ 2

End of examination