



YEAR 12 MATHEMATICS

EXTENSION 2

TRIAL EXAMINATION 2008

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- There are 8 questions.
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working must be shown in every question
- All questions are of equal value
- Use a separate booklet for each question
- A table of standard integrals is provided at the back of this paper.

Question 1 (15 marks)

Marks

(a) Find $\int \frac{x}{(2-x)^3} dx$ 2

(b) By completing the square, find $\int \frac{dx}{\sqrt{6x-x^2}}$ 2

(c) Use integration by parts to find $\int_0^{\frac{\pi}{2}} x \cos 2x dx$ 3

(d) (i) Find values for A, B and C so that 2

$$\frac{4x^2 + 7x - 5}{(x-2)(x+3)^2} \equiv \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

(ii) Hence find $\int \frac{4x^2 + 7x - 5}{(x-2)(x+3)^2} dx$ 2

(e) Use the substitution $t = \tan \frac{x}{2}$ to evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1 - \cos x} dx$ 4

Question 2 (12 marks) – Use a SEPARATE writing booklet

Marks

(a) Let $z = 1 + 3i$ and $w = 1 - 2i$

Find, in the form $x + iy$,

(i) $(z + w)^2$

1

(ii) $z\bar{w}$

1

(iii) $\frac{z}{w}$

1

(b) Let $z = -\sqrt{3} - i$

(i) Express z in modulus-argument form

2

(ii) Hence evaluate z^{10} in the form $x + iy$.

2

(c) On an Argand diagram, shade the region where the inequalities

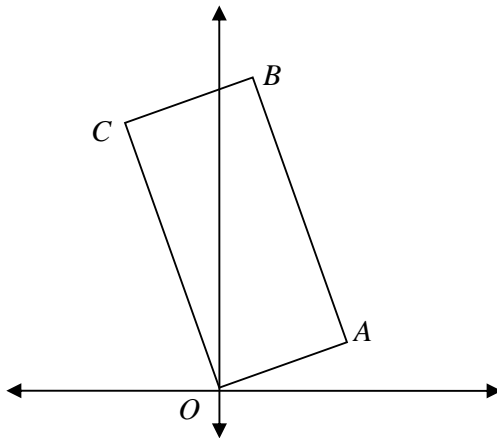
3

$$|z - 3 - 2i| \leq 2 \quad \text{and} \quad -\frac{\pi}{2} < \arg(z - 3 - 2i) < \frac{\pi}{4} \quad \text{both hold}$$

(d) Find, in the form $x + iy$, complex numbers such that $(x + iy)^2 = 16 + 30i$

3

(e)



OABC is a rectangle in which $OC = 2OA$

The point A represents the complex number z

(i) What is the complex number represented by the point C

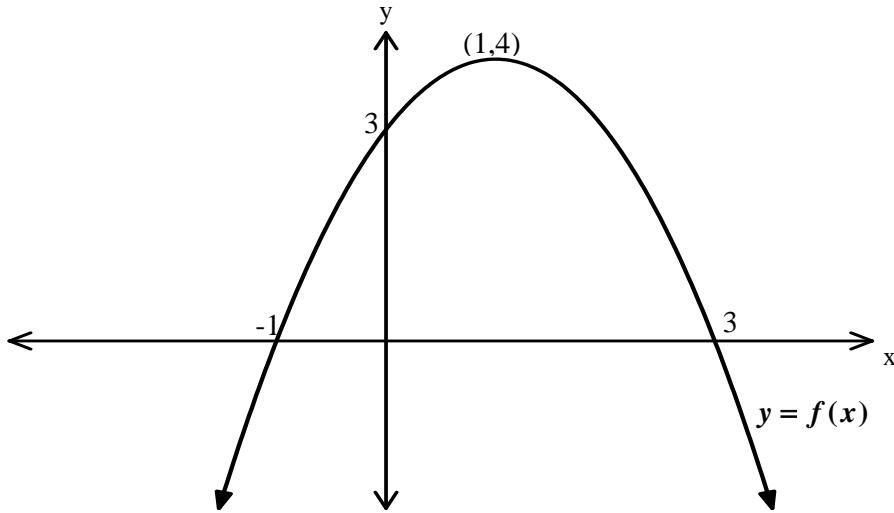
1

(ii) What is the complex number that is represented by \vec{CA}

1

Question 3 (15 marks) Use a SEPARATE writing booklet

Marks



(a) The diagram shows the graph of $y = f(x)$

Make separate, one-third page sketches showing the main features of the graphs of

(i) $y = f(-x)$ 2

(ii) $y = (f(x))^2$ 2

(iii) $y^2 = f(x)$ 2

(iv) $y = \frac{1}{f(x)}$ 2

(b) Determine the equation of the normal to the curve $x^2 - 2xy - y^2 = 2$ at the point (3,1) on the curve. 3

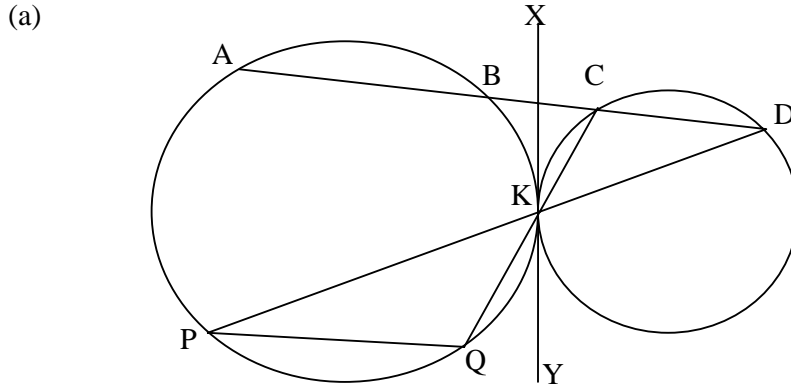
(c) Consider the graph of the function whose equation is $y = x + \frac{5x}{x^2 - 4}$ 2

(i) Write down the equations of asymptotes and the coordinates of the intercepts with the coordinate axes. 2

(ii) Make a neat half-page sketch of the graph of $y = x + \frac{5x}{x^2 - 4}$

Question 4 (15 marks) Use a SEPARATE writing booklet

Marks



The diagram shows two different circles that touch externally at K.
 XY is a common tangent to the two circles at K.
 The straight line ABCD cuts the first circle at A and B and cuts the second circle at C and D.
 The straight line through D and K cuts the first circle at P.
 The straight line through C and K cuts the first circle at Q.

4

Copy or trace the diagram into your examination booklet.

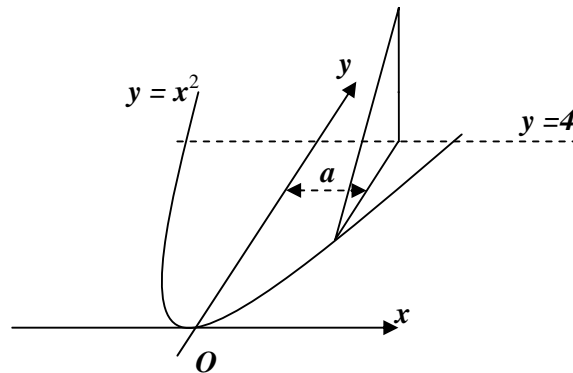
Prove that PQ is parallel to AD

- (b) The roots of $x^3 - 3x^2 - 2x + 4 = 0$ are α, β and γ **2**
- (i) Find a cubic polynomial equation with integer coefficients whose roots are α^2, β^2 and γ^2 **1**
- (ii) Hence, or otherwise, find the value of $\alpha^2 + \beta^2 + \gamma^2$ **2**
- (iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$ **3**
- (c) Let $P(x) = x^3 + 3x^2 - 24x + k$ **3**
- Find the possible values of k given that the equation $P(x) = 0$ has a double root.
- (d) When the polynomial $P(x)$ is divided by $(x+2)(x-3)$, the remainder is $4x+1$. **3**
- Find the remainder when $P(x)$ is divided by $(x+2)$

Question 5 (15 marks) Use a SEPARATE writing booklet

Marks

(a)



The base of a solid is the region bound by the parabola $y = x^2$ and the line $y = 4$. Vertical cross-sections parallel to the y -axis are right-angled isosceles triangles with the right-angle on the line $y = 4$.

(i) Show that the area of a typical cross-section, at a distance a units

2

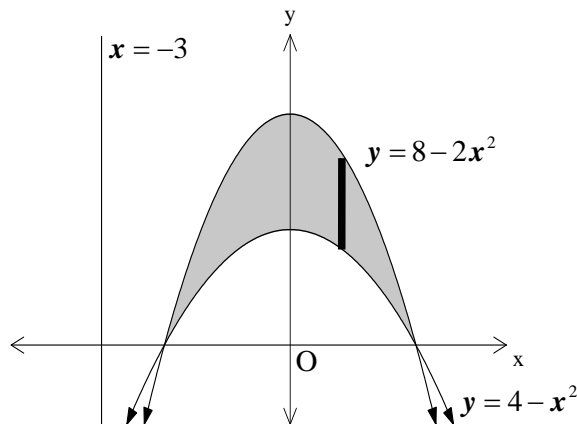
from the y -axis is given by $A = \frac{1}{2}(a^4 - 8a^2 + 16)$.

(ii) Form an integral whose value will give the volume of the solid.

3

Evaluate this integral to find the volume of the solid.

(b)



The region bound by $y = 8 - 2x^2$ and $y = 4 - x^2$ is rotated about the line $x = -3$.

Use the method of cylindrical shells to find the volume of the solid that is formed.

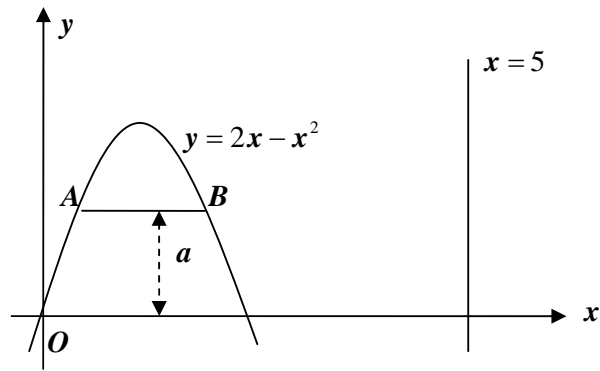
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Question 5 continues on the next page.

Question 5 (continued)

Marks

(c)



The region bound by $y = 2x - x^2$ and the x -axis is to be rotated about the line $x = 5$ to form a solid.

(i) AB is a horizontal chord of $y = 2x - x^2$, at a height a units above the x -axis.

When this chord is rotated about the line $x = 5$, it will form an annulus.

Show that the area of this annulus is given by $A = 16\pi\sqrt{1-y}$

3

(ii) Form an integral whose value will give the volume of the solid.

Hence find the volume of the solid.

2

Question 6 (15 marks) Use a SEPARATE writing booklet

Marks

(a) Consider the ellipse whose equation is $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(i) Find the eccentricity of the ellipse

1

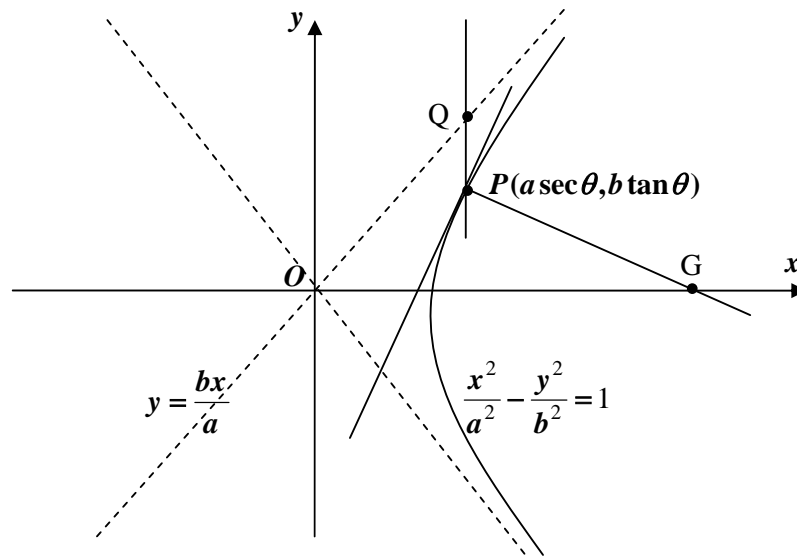
(ii) Find the coordinates of the foci and the equations of the directrices

2

(iii) Make a neat sketch of the ellipse clearly showing and labelling the foci and directrices.

2

(b)



$P(a \sec \theta, b \tan \theta)$ is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The equation of the normal at $P(a \sec \theta, b \tan \theta)$ is given by

$$\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2 \quad (\text{Do not prove this})$$

A line through P parallel to the y -axis meets the asymptote $y = \frac{bx}{a}$ at Q .

The normal at P meets the x -axis at G .

(i) Find the coordinates of Q and G

2

(ii) Show that $\angle OQG = 90^\circ$, where O is the origin.

2

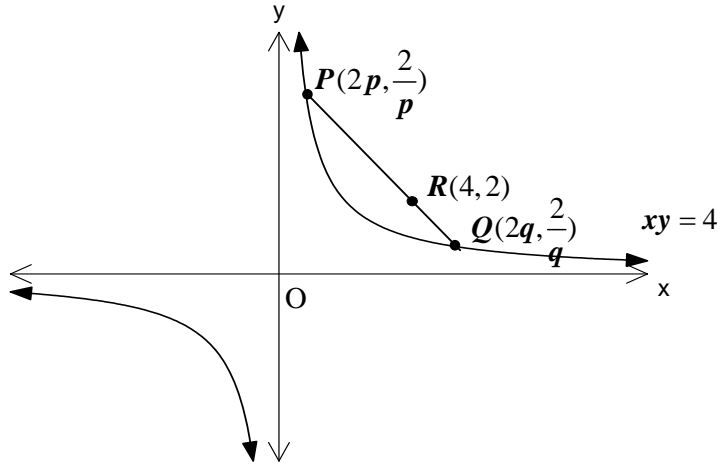
Question 6 continues on the next page

Question 6 (Continued)

Marks

(c) $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ are two points on the rectangular hyperbola $xy = 4$.

The chord PQ always passes through the point $R(4,2)$



(i) Show that the equation the chord PQ is $x + pqy = 2(p + q)$ **2**

(ii) Show that $pq = p + q - 2$ **1**

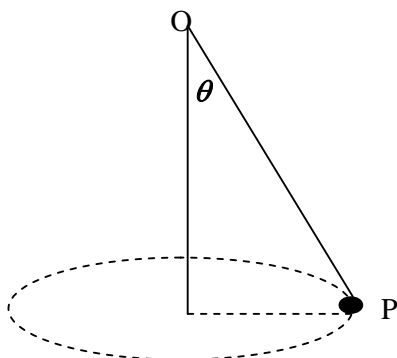
(iii) Let M be the midpoint of PQ **3**

Write down the coordinates of M and hence find the equation of the locus of M as the points P and Q move on the curve $xy = 4$.

Question 7 (15 marks) – Use a SEPARATE writing booklet

Marks

- (a) A body P of mass 0.5 kg is suspended from a fixed point O by a light, inextensible string of length 1 (one) metre. The mass is rotated in a horizontal circle with constant speed of v metres per second. The string makes an angle of θ° with the downward direction of the vertical.



- (i) Copy the diagram onto your own paper and show the forces that are acting on P. 1
- (ii) By resolving the horizontal and vertical forces acting on P, show that $\tan \theta = \frac{v^2}{rg}$ where r is the radius of the circle. 3

For parts (iii) and (iv) assume $g = 9.8$ and $\theta = 30^\circ$

- (iii) Find the tension in the string 1
- (iv) Find the speed v of P 1
- (b) A rock of mass 5 kg is propelled vertically upwards into the air from the ground with an initial velocity of 12 ms^{-1} . The rock is subject to a downward gravitational force of 50 Newtons (ie $mg = 50$) and air resistance of $\frac{v^2}{2}$ Newtons in the opposite direction to the velocity, $v \text{ ms}^{-1}$ 1
- (i) Make a neat sketch showing the forces acting and the rock. 1
- Hence show that the equation of motion of the rock is $\ddot{x} = -\frac{v^2}{10} - 10$
- (ii) Using $\ddot{x} = \frac{dv}{dt}$, find the time taken for the rock to reach its maximum height. 3
- (iii) Using $\ddot{x} = v \frac{dv}{dx}$, show that $v^2 = 244e^{-\frac{x}{5}} - 100$ 3
- (iv) Find the maximum height reached by the rock. 2

Question 8 (15 marks) – Use a SEPARATE writing booklet

- | | Marks |
|--|--------------|
| (a) Use the principle of mathematical induction to prove that
$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$ for all integers $n \geq 1$ | 4 |
| (b) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$ for $n = 0, 1, 2, \dots$ | 3 |
| (i) Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n = 2, 3, 4, \dots$ | 2 |
| (ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x dx$ | |
| (c) (i) On an Argand diagram, plot and label the 5 points that represent the roots of the equation $z^5 - 1 = 0$. | 2 |
| (ii) Hence express $z^5 - 1$ as a product of one linear and two quadratic factors with real coefficients. | 2 |
| (iii) Using the result $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$ and part (i), write down the roots of $z^4 + z^3 + z^2 + z + 1 = 0$ | 1 |
| (iv) Hence show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$ | 1 |

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, x > 0$