YEAR 12 MATHEMATICS

EXTENSION 2

TRIAL EXAMINATION 2008

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- There are 8 questions.
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working must be shown in every question
- All questions are of equal value
- Use a separate booklet for each question
- A table of standard integrals is provided at the back of this paper.

<u>Question 1</u> (15 marks)

Marks

(a) Find $\displaystyle\int \frac{x}{(2-x)^3}\,dx$ **2**

(b) By completing the square, find $\displaystyle\int \frac{dx}{\sqrt{6x-x^2}}\,dx$ **2**

(c) Use integration by parts to find $\displaystyle\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$ **3**

(d) (i) Find values for A, B and C so that **2**

$$\frac{4x^2+7x-5}{(x-2)(x+3)^2} \equiv \frac{A}{x-2}+\frac{B}{(x+3)}+\frac{C}{(x+3)^2}$$

(ii) Hence find $\displaystyle\int \frac{4x^2+7x-5}{(x-2)(x+3)^2}\,dx$ **2**

(e) Use the substitition $t=\tan\frac{x}{2}$ to evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\,dx$ **4**

<u>Question 2</u> (12 marks) – Use a SEPARATE writing booklet

Marks

(a) Let $\boldsymbol{z} = 1+3\boldsymbol{i}$ and $\boldsymbol{w} = 1-2\boldsymbol{i}$

Find, in the form $\boldsymbol{x}+\boldsymbol{iy}$,

(i) $(\boldsymbol{z}+\boldsymbol{w})^2$ **1**

(ii) $\boldsymbol{z}\overline{\boldsymbol{w}}$ **1**

(iii) $\dfrac{\boldsymbol{z}}{\boldsymbol{w}}$ **1**

(b) Let $\boldsymbol{z} = -\sqrt{3}-\boldsymbol{i}$

(i) Express $\boldsymbol{z}$ in modulus-argument form **2**

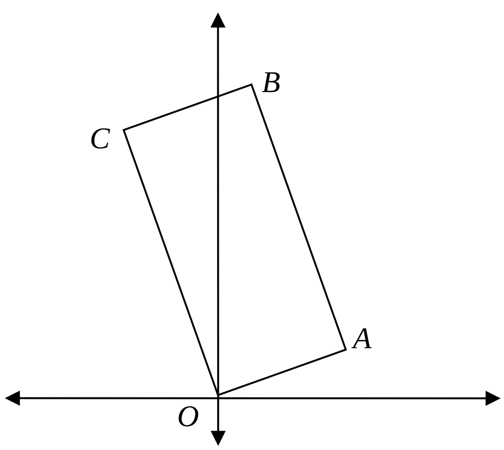
(ii) Hence evaluate $\boldsymbol{z}^{10}$ in the form $\boldsymbol{x}+\boldsymbol{iy}$. **2**

(c) On an Argand diagram, shade the region where the inequalities **3**

$|\boldsymbol{z}-3-2\boldsymbol{i}| \le 2$ and $-\dfrac{\pi}{2} < \arg(\boldsymbol{z}-3-2\boldsymbol{i}) < \dfrac{\pi}{4}$ both hold

(d) Find, in the form $\boldsymbol{x}+\boldsymbol{iy}$, complex numbers such that $(\boldsymbol{x}+\boldsymbol{iy})^2 = 16+30\boldsymbol{i}$ **3**

(e)



OABC is a rectangle in which $\boldsymbol{OC} = 2\,\boldsymbol{OA}$

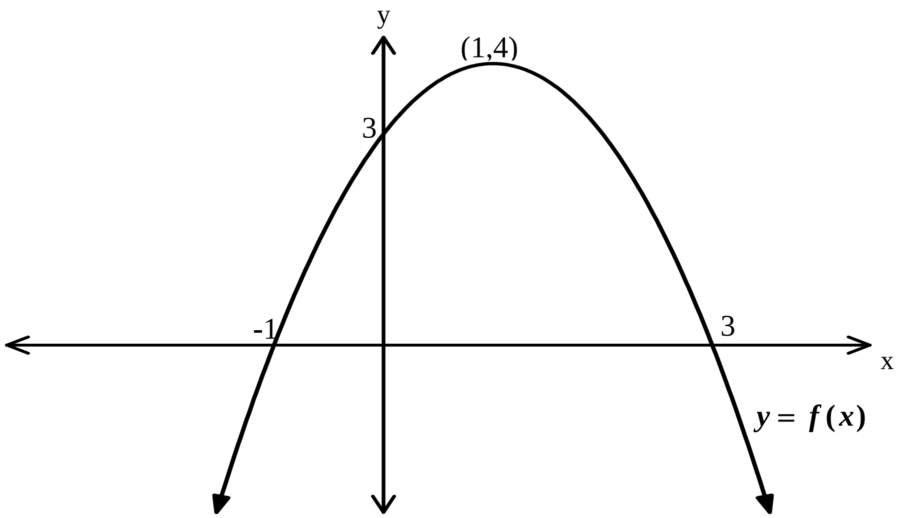
The point $\boldsymbol{A}$ represents the complex number $\boldsymbol{z}$

(i) What is the complex number represented by the point C **1**

(ii) What is the complex number that is represented by $\overrightarrow{\mathrm{CA}}$ **1**

<u>Question 3</u> (15 marks) Use a SEPARATE writing booklet

Marks



(a) The diagram shows the graph of $y = f(x)$

Make separate, one-third page sketches showing the main features of the graphs of

(i) $y = f(-x)$ **2**

(ii) $y = \left(f(x)\right)^2$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(b) Determine the equation of the normal to the curve $x^2 - 2xy - y^2 = 2$ at the point (3,1) on the curve. **3**

(c) Consider the graph of the function whose equation is $y = x + \dfrac{5x}{x^2 - 4}$

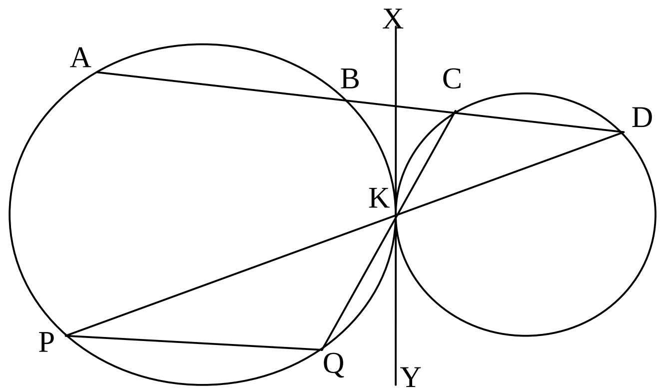
(i) Write down the equations of asymptotes and the coordinates of the intercepts with the coordinate axes. **2**

(ii) Make a neat half-page sketch of the graph of $y = x + \dfrac{5x}{x^2 - 4}$ **2**

Question 4 (15 marks) Use a SEPARATE writing booklet

Marks

(a)



The diagram shows two different circles that touch externally at K.
XY is a common tangent to the two circles at K.
The straight line ABCD cuts the first circle at A and B and cuts the second circle at C and D.
The straight line through D and K cuts the first circle at P.
The straight line through C and K cuts the first circle at Q.

Copy or trace the diagram into your examination booklet. **4**

Prove that PQ is parallel to AD

(b) The roots of $x^3 - 3x^2 - 2x + 4 = 0$ are α, β and γ

(i) Find a cubic polynomial equation with integer coefficients whose roots are α^2, β^2 and γ^2 **2**

(ii) Hence , or otherwise, find the value of $\alpha^2 + \beta^2 + \gamma^2$ **1**

(iii) Detemine the value of $\alpha^3 + \beta^3 + \gamma^3$ **2**

(c) Let $P(x) = x^3 + 3x^2 - 24x + k$

Find the possible values of k given that the equation $P(x) = 0$ has a double root. **3**

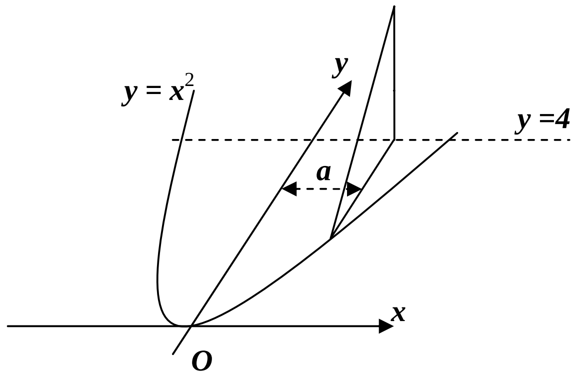
(d) When the polynomial $P(x)$ is divided by $(x+2)(x-3)$, the remainder is $4x+1$. **3**

Find the remainder when $P(x)$ is divided by $(x+2)$

<u>Question 5</u> (15 marks) Use a SEPARATE writing booklet

Marks

(a)

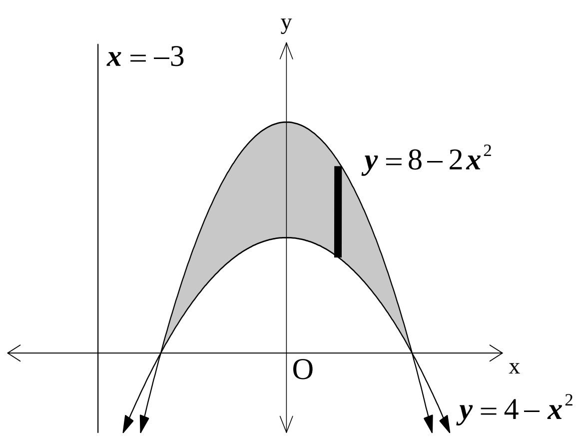


The base of a solid is the region bound by the parabola $y = x^2$ and the line $y = 4$. Vertical cross-sections parallel to the y-axis are right-angled isosceles triangles with the right-angle on the line $y = 4$.

(i) Show that the area of a typical cross-section, at a distance a units from the y-axis is given by $A = \frac{1}{2}(a^4 - 8a^2 + 16)$. **2**

(ii) Form an integral whose value will give the volume of the solid. Evaluate this integral to find the volume of the solid. **3**

(b)

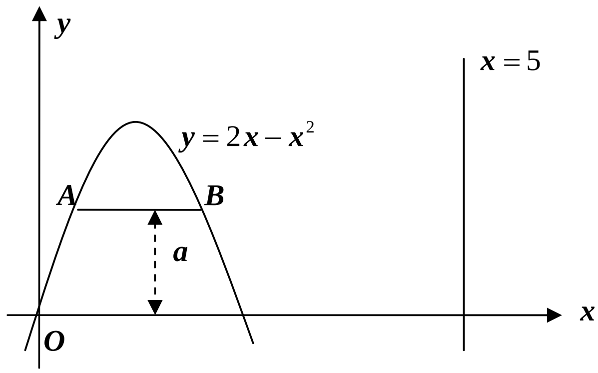


The region bound by $y = 8 - 2x^2$ and $y = 4 - x^2$ is rotated about the line $x = -3$.
Use the method of cylindrical shells to find the volume of the solid that is formed. **5**

Question 5 continues on the next page.

Marks

(c)



The region bound by $y = 2x - x^2$ and the x – axis is to be rotated about the line $x = 5$ to form a solid.

(i) AB is a horizontal chord of $y = 2x - x^2$, at a height a units above the x-axis. When this chord is rotated about the line $x = 5$, it will form an annulus. Show that the area of this annulus is given by A = $16\pi\sqrt{1-y}$ **3**

(ii) Form an integral whose value will give the volume of the solid. Hence find the volume of the solid. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet

Marks

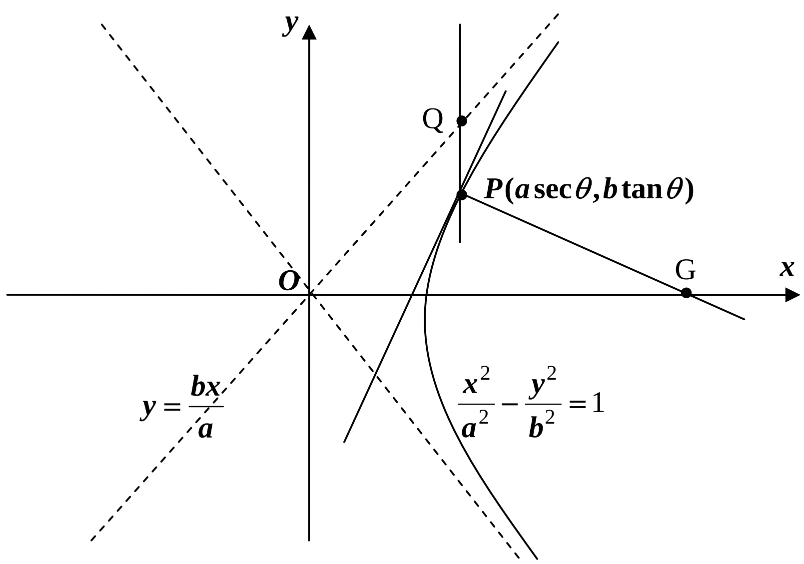
(a) Consider the ellipse whose equation is $\frac{x^2}{16}+\frac{y^2}{9}=1$

(i) Find the eccentricity of the ellipse **1**

(ii) Find the coordinates of the focii and the equations of the directrices **2**

(iii) Make a neat sketch of the ellipse clearly showing and labelling the focii and directrices. **2**

(b)



$P(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The equation of the normal at $P(a\sec\theta, b\tan\theta)$ is given by

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2 \quad \text{(Do not prove this)}$$

A line through P parallel to the y-axis meets the asymptote $y=\frac{bx}{a}$ at Q.

The normal at P meets the x-axis at G.

(i) Find the coordinates of Q and G **2**

(ii) Show that $\angle OQG = 90^\circ$, where O is the origin. **2**

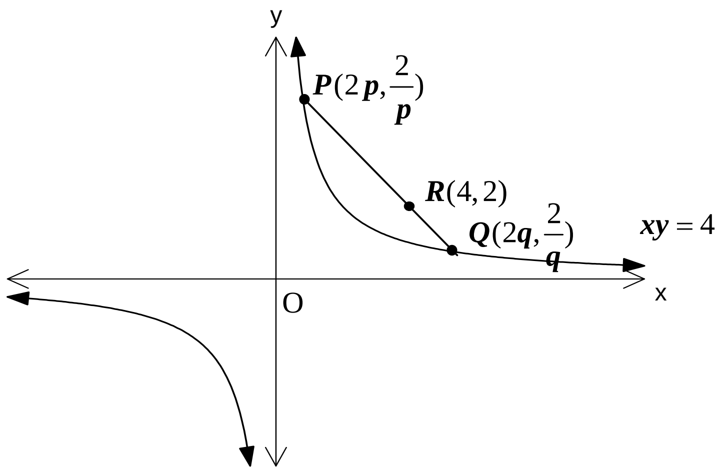
Question 6 continues on the next page

Question 6 (Continued)

Marks

(c) $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ are two points on the rectangular hyperbola $xy = 4$.

The chord PQ always passes through the point $R(4, 2)$



(i) Show that the equation the chord PQ is $x + pqy = 2(p+q)$ **2**

(ii) Show that $pq = p + q - 2$ **1**

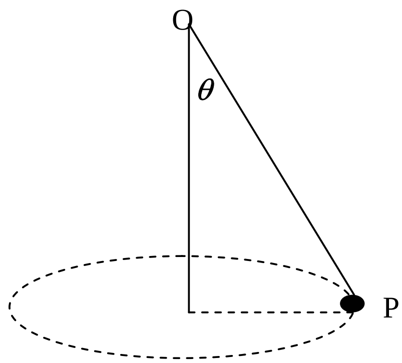
(iii) Let M be the midpoint of PQ

Write down the coordinates of M and hence find the equation of the locus of M as the points P and Q move on the curve $xy = 4$. **3**

<u>Question 7</u> (15 marks) – Use a SEPARATE writing booklet

Marks

(a) A body P of mass 0.5 kg is suspended from a fixed point O by a light , inextensible string of length 1 (one)metre. The mass is rotated in a horizontal circle with constant speed of v metres per second. The string makes an angle of θ^o with the downward direction of the vertical.



(i) Copy the diagram onto your own paper and show the forces that are acting on P. **1**

(ii) By resolving the horizontal and vertical forces acting on P, show that

$\tan\theta = \dfrac{\boldsymbol{v}^2}{\boldsymbol{rg}}$ where $\boldsymbol{r}$ is the radius of the circle. **3**

For parts (iii) and (iv) assume $\boldsymbol{g} = 9.8$ and $\theta = 30^o$

(iii) Find the tension in the string **1**

(iv) Find the speed $\boldsymbol{v}$ of P **1**

(b) A rock of mass 5 kg is propelled vertically upwards into the air from the ground with an initial velocity of 12 ms^{-1}. The rock is subject to a downward gravitational force of 50 Newtons (ie $\boldsymbol{mg}$ = 50) and air resistance of $\dfrac{\boldsymbol{v}^2}{2}$ Newtons in the opposite direction to the velocity, v ms^{-1}

(i) Make a neat sketch showing the forces acting and the rock. **1**

Hence show that the equation of motion of the rock is $\ddot{\boldsymbol{x}} = -\dfrac{\boldsymbol{v}^2}{10} - 10$

(ii) Using $\ddot{\boldsymbol{x}} = \dfrac{\boldsymbol{dv}}{\boldsymbol{dt}}$, find the time taken for the rock to reach its maximum height. **3**

(iii) Using $\ddot{\boldsymbol{x}} = \boldsymbol{v}\dfrac{\boldsymbol{dv}}{\boldsymbol{dx}}$, show that $\boldsymbol{v}^2 = 244\boldsymbol{e}^{-\frac{x}{5}} - 100$ **3**

(iv) Find the maximum height reached by the rock. **2**

<u>Question 8</u> (15 marks) – Use a SEPARATE writing booklet

Marks

(a) Use the principle of mathematical induction to prove that **4**

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all integers $n \geq 1$

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$ for $n = 0, 1, 2, \ldots\ldots$ **3**

(i) Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n = 2, 3, 4 \ldots.$ **2**

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx$

(c) (i) On an Argand diagram, plot and label the 5 points that represent the roots of the equation $z^5 - 1 = 0$. **2**

(ii) Hence express $z^5 - 1$ as a product of one linear and two quadratic factors with real coefficients. **2**

(iii) Using the result $z^5 - 1 = (z-1)(z^4 + z^3 + z^3 + z + 1)$ and part (i), write down the roots of $z^4 + z^3 + z^3 + z + 1 = 0$ **1**

(iv) Hence show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ **1**

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \quad , n \neq -1 \ ; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x \quad , x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} \quad , a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax \quad , \ a \neq 0$$

$$\int \sin ax \, dx = -\tfrac{1}{a} \cos ax \, , a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax \ , \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax \ , a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \ , \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \ , \ a > 0 \ , \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}\,) \ , x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}\,)$$

NOTE: $\ln x = \log_e x \ , \ x > 0$