# Sydney Grammar School

# 4 unit mathematics

## Trial HSC Examination 1999

**1. (a)** Let $z = \frac{1-i}{2+i}$.
**(i)** Show that $z + \frac{1}{z} = \frac{3+2i}{3-i}$.
**(ii)** Hence find **($\alpha$)** $\overline{(z + \frac{1}{z})}$, in the form $a + ib$, where $a$ and $b$ are real,
**($\beta$)** $\Im(z + \frac{1}{z})$.
**(b) (i)** Express $-\sqrt{27} - 3i$ in modulus-argument form.
**(ii)** Hence find $(-\sqrt{27} - 3i)^6$, giving your answer in the form $a + bi$, where $a$ and $b$ are real.
**(c)** Sketch on separate Argand diagrams the locus of $z$ defined as follows:
**(i)** $\arg(z-1) = \frac{3\pi}{4}$ **(ii)** $\Re\big(z(\overline{z}+2)\big) = 3$.
**(d)** If $z$ is a complex number such that $z = k(\cos\phi + i\sin\phi)$, where $k$ is real, show that $\arg(z+k) = \frac{1}{2}\phi$.

**2. (a)** Find $\int x^2 e^{-2x}\, dx$.
**(b) (i)** Resolve $\frac{9+x-2x^2}{(1-x)(3+x^2)}$ into partial fractions.
**(ii)** Hence find $\int \frac{9+x-2x^2}{(1-x)(3+x^2)}\, dx$.
**(c)** Evaluate each of the following:
**(i)** $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$,
**(ii)** $\int_0^{\frac{\pi}{3}} \sec^4\theta \tan\theta\, d\theta$,
**(iii)** $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta+\cos\theta}\, d\theta$. (Hint: Use the substitution $t = \tan\frac{\theta}{2}$.)

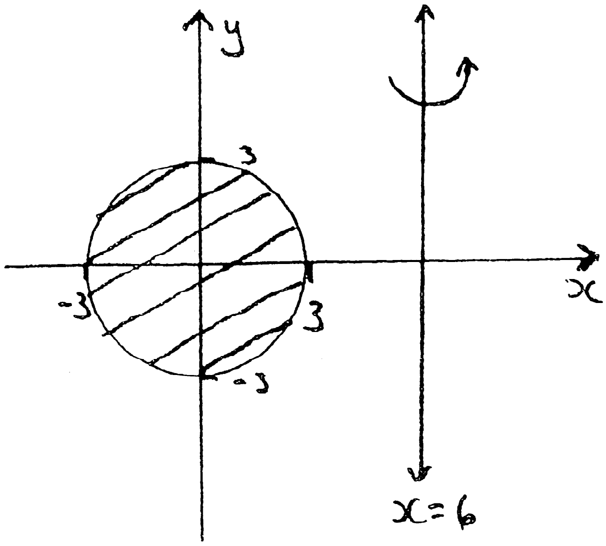
**3. (a)** Consider the function $y = \ln(\ln x)$.
**(i)** State the domain of the function.
**(ii)** Prove that the function is increasing at all points in its domain.
**(iii)** On separate number planes, sketch the following, clearly labelling all axial intercepts and asymptotes:
**($\alpha$)** $y = \ln(\ln x)$, **($\beta$)** $y = \ln(\ln|x|)$, **($\gamma$)** $\ln|\ln x|$.
**(b)** Find a cubic equation with roots $\alpha, \beta$ and $\gamma$ such that:

$$\begin{aligned} \alpha\beta\gamma &= 5, \text{ and} \\ \alpha+\beta+\gamma &= 7, \text{ and} \\ \alpha^2+\beta^2+\gamma^2 &= 29. \end{aligned}$$

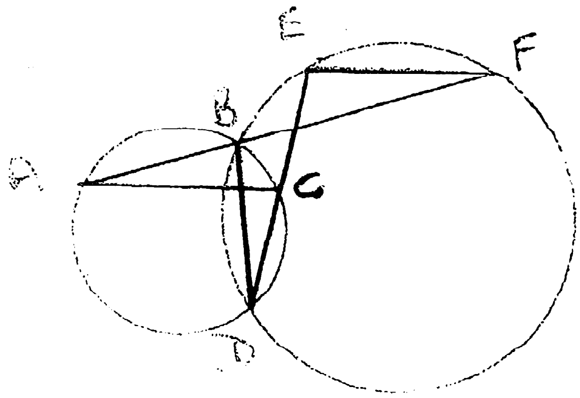
**(c)** If $\alpha, \beta$ and $\gamma$ are the roots of the equation $8x^3 - 4x^2 + 6x - 1 = 0$, find the equation whose roots are $\frac{1}{1-\alpha}$, $\frac{1}{1-\beta}$ and $\frac{1}{1-\gamma}$.

**(d)** If the equation $x^3 + 3kx + \ell = 0$ has a double root, where $k$ and $\ell$ are real, prove that $\ell^2 = -4k^3$.

**4. (a) (i)** Prove that the function $f(x) = x\sqrt{a^2 - x^2}$ is odd.




**(ii)** The diagram shows the region $x^2 + y^2 \leq 9$ and the line $x = 6$. Copy the diagram.
**(iii)** Use the method of cylindrical shells to show that if the region $x^2 + y^2 \leq 9$ is rotated about the line $x = 6$ the volume $V$ of the torus formed is given by $V = 24\pi \int_{-3}^{3} \sqrt{9 - x^2}\, dx - 4\pi \int_{-3}^{3} x\sqrt{9 - x^2}\, dx$.
**(iv)** Hence find the volume of the torus.
**(b)**

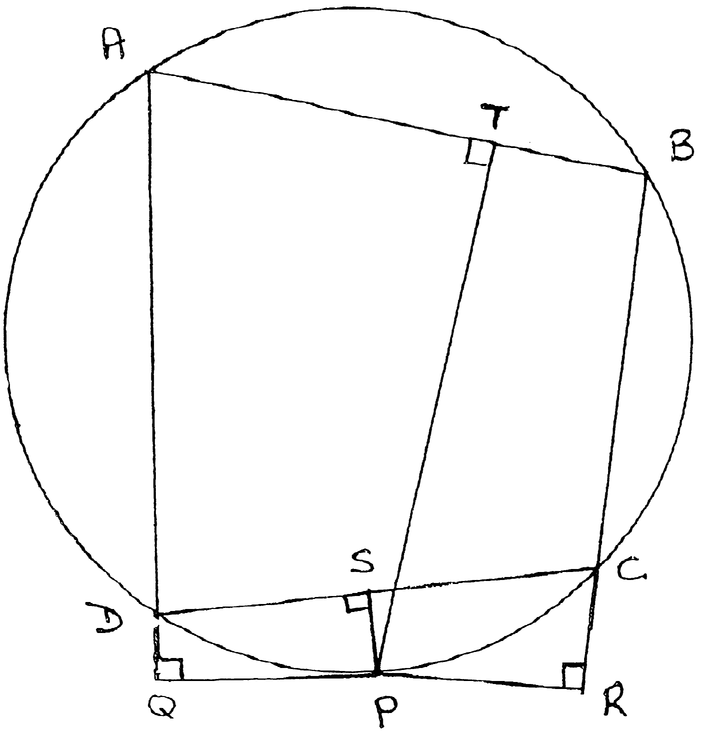



In the diagram above, $ABF$ and $DCE$ are straight lines.
**(i)** Copy the diagram.
**(ii)** Prove that $AC$ is parallel to $EF$.
**(c)** A car of mass 1.8 tonnes negotiates a curve of radius 130 metres, banked at an angle of 9° to the horizontal, at a constant speed of 70 km/h. Take the acceleration due to gravity to be $10\text{m/s}^2$.

**(i)** Draw a diagram showing all the forces acting on the car.
**(ii)** By resolving forces vertically and horizontally, calculate the frictional force between the road surface and the wheels, to the nearest Newton.
**(iii)** What speed (to the nearest km/h) must the driver maintain in order for the car to experience no sideways frictional force?

**5. (a)** A particle of mass $m$ projected vertically upwards with initial speed $u$ metres per second experiences a resistance of magnitude $kmv$ Newtons when the speed is $v$ metres per second where $k$ is a positive constant. After $T$ seconds the particle attains its maximum height $h$. Let the acceleration due to gravity be $g$ m/s$^2$.
**(i)** Show that the acceleration of the particle is given by $\ddot{x} = -(g + kv)$, where $x$ is the height of the particle $t$ seconds after the launch.
**(ii)** Prove that $T$ is given by $T = \frac{1}{k}\ln(\frac{g+ku}{g})$ seconds.
**(iii)** Prove that $h$ is given by $h = \frac{u-gT}{k}$ metres.
**(b)** Let $A$ and $B$ be the points $(1, 1)$ and $(b, \frac{1}{b})$ respectively, where $b > 1$.
**(i)** The tangents to the curve $y = \frac{1}{x}$ at $A$ and $B$ intersect at $C(\alpha, \beta)$. Show that $\alpha = \frac{2b}{b+1}$ and $\beta = \frac{2}{b+1}$.
**(ii)** Let $A', B'$ and $C'$ be the points $(1, 0), (b, 0)$ and $(\alpha, 0)$ respectively.
**($\boldsymbol{\alpha}$)** Draw a diagram that represents the information above.
**($\boldsymbol{\beta}$)** Obtain an expression for the sum of the areas of the quadrilaterals $ACC'A'$ and $CBB'C'$.
**($\boldsymbol{\gamma}$)** Hence or otherwise prove that for $u > 0$, $\frac{2u}{2+u} < \ln(1 + u) < u$.

**6. (a)**




In the diagram above, $ABCD$ is a cyclic quadrilateral. $P$ is a point on the circle through $A, B, C$ and $D$. $PQ, PR, PS$ and $PT$ are the perpendiculars from $P$ to $AD$ produced, $BC$ produced, $CD$ and $AB$ respectively.

**(i)** Copy the diagram.

**(ii)** Explain why $SPRC$ and $AQPT$ are cyclic quadrilaterals.

**(iii)** Hence show that $\angle SPR = \angle QPT$ and $\angle PRS = \angle PTQ$.

**(iv)** Prove that $\triangle SPR$ is similar to $\triangle QPT$.

**(v)** Hence show that **(α)** $PS \times PT = PQ \times PR$, **(β)** $\frac{PS \times PR}{PQ \times PT} = \frac{SR^2}{QT^2}$.

**(b)** The function $f$ is given by $f(x) = e^{x/(1+kx)}$, where $k$ is a positive constant.

**(i)** Find $f'(x)$ and $f''(x)$.

**(ii)** Show $f(x)$ has a point of inflexion at $(\frac{1}{2k^2} - \frac{1}{k}, e^{\frac{1}{k}-2})$.

**(iii)** Show that the tangent to $y = f(x)$ at $x = a$ passes through the origin if and only if $k^2a^2 + (2k-1)a + 1 = 0$.

**(iv)** Hence show that no tangents to $y = f(x)$ pass through the origin if and only if $k > \frac{1}{4}$.

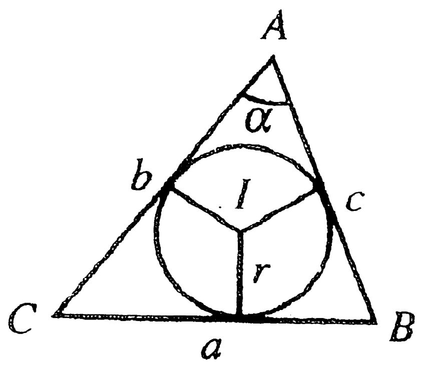
**7. (a)** Let $P(z) = z^7 - 1$.

**(i)** Solve the equation $P(z) = 0$, displaying your seven solutions on an Argand diagram.

**(ii)** Show that $P(z) = z^3(z-1)\big((z+\frac{1}{z})^3 + (z+\frac{1}{z})^2 - 2(z+\frac{1}{z}) - 1\big)$.

**(iii)** Hence solve the equation $x^3 + x^2 - 2x - 1 = 0$.

**(iv)** Hence prove that $\operatorname{cosec}\frac{\pi}{14}\operatorname{cosec}\frac{3\pi}{14}\operatorname{cosec}\frac{5\pi}{14} = 8$.
**(b)**




The diagram above shows a circle, centre $I$ and radius $r$, touching the three sides of a triangle $ABC$. Denote $AB$ by $c$, $BC$ by $a$ and $AC$ by $b$. Let $\angle BAC = \alpha$, $s = \frac{1}{2}(a+b+c)$ and $\Delta =$ the area of triangle $ABC$.
**(i)** By considering the area of the triangles $AIB, BIC$ and $CIA$, or otherwise, show that $\Delta = rs$.
**(ii)** By using the formula $\Delta = \frac{1}{2}bc\sin\alpha$, show that $\Delta^2 = \frac{1}{16}(4b^2c^2 - (2bc\cos\alpha)^2)$.
**(iii)** Use the cosine rule to show that $\Delta^2 = \frac{1}{16}(a^2-(b-c)^2)((b+c)^2-a^2)$. Hence deduce that $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.
**(iv)** A hole in the shape of the triangle $ABC$ is cut in the top of a level table. A sphere of radius $R$ rests in the hole. Find the height of the centre of the sphere above the level of the table-top, expressing your answer in terms of $a, b, c, s$ and $R$.

**8. (a)** For $n = 0, 1, 2, 3, \ldots$, define $I_n = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2nx}{\sin 2x}\, dx$.
**(i)** Evaluate $I_1$.
**(ii)** Show that, for $r \geq 1$:
**($\alpha$)** $I_{2r+1} - I_{2r-1} = \frac{1-(-1)^r}{2r}$.
**($\beta$)** $I_{2r} - I_{2r-2} = \frac{1}{2r-1}$.
**(iii)** Hence evaluate $I_8$ and $I_9$.
**(b)** The Bernoulli polynomials $B_n(x)$, are defined by $B_0(x) = 1$ and, for $n = 1, 2, 3, \ldots$, $\frac{dB_n(x)}{dx} = nB_{n-1}(x)$, and $\int_0^1 B_n(x)\, dx = 0$. Thus

$$B_1(x) = x - \tfrac{1}{2},$$
$$B_2(x) = x^2 - x + \tfrac{1}{6},$$
$$B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x.$$

**(i)** Show that $B_4(x) = x^2(x-1)^2 - \frac{1}{30}$.
**(ii)** Show that, for $n \geq 2$, $B_n(1) - B_n(0) = 0$.
**(iii)** Show, by mathmatical induction, that for $n \geq 1$: $B_n(x+1) - B_n(x) = nx^{n-1}$.
**(iv)** Hence show that for $n \geq 1$ and any positive integer $k$:
$n\sum_{m=0}^{k} m^{n-1} = B_n(k+1) - B_n(0)$.
**(v)** Hence deduce that $\sum_{m=0}^{135} m^4 = 9\ 134\ 962\ 308$.