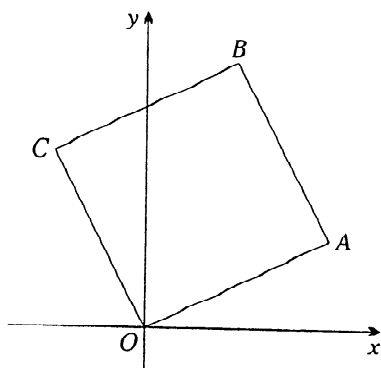


Sydney Grammar School

4 unit mathematics

Trial HSC Examination 2000

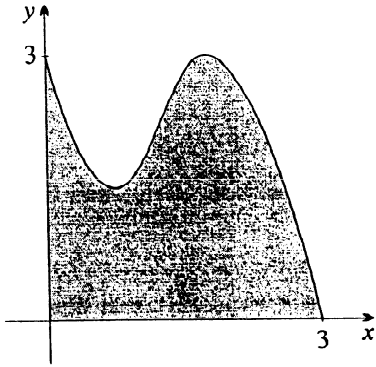
1. (a) Evaluate $\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}} dx$.
- (b) The integral I_n is defined by $I_n = \int_0^1 x^n e^{-x} dx$.
- (i) Show that $I_n = nI_{n-1} - e^{-1}$.
- (ii) Hence show that $I_3 = 6 - 16e^{-1}$.
- (c) Use partial fractions to find $\int \frac{2y+3}{(y-2)(y^2+3)} dy$.
- (d) Use integration by parts to find $\int \tan^{-1} x dx$.
- (e) (i) Find $\int \frac{dx}{x^2+2x+5}$.
- (ii) Hence find $\int \frac{x^2}{x^2+2x+5} dx$.
2. (a) Simplify $(2 - 3i)^2$.
- (b) On an Argand diagram, sketch the region specified by both the conditions $|z + 3 - 4i| \leq 5$ and $\Re(z) \leq 1$. You must show intercepts with the axes, but you do not need to find other points of intersection.
- (c) (i) Determine the modulus and argument of $-1 + i$.
- (ii) Hence find the least positive value of n for which $(-1 + i)^n$ is real.
- (d) (i) Let $z = r(\cos \theta + i \sin \theta)$ be a complex number in the Argand diagram. Show that multiplication of z by i rotates z by $\frac{\pi}{2}$ anticlockwise about the origin.
- (ii)



In the square $OABC$, shown above, the point A represents $2 + i$. What complex numbers do the points B and C represent?

- (e) Let $z = a + ib$, where a and b are both real.
- (i) For what values of a and b will $z + \frac{1}{z}$ be purely real?
- (ii) Is it possible for $z + \frac{1}{z}$ to be purely imaginary?

3. (a)



In the diagram above, the region under the curve $y = 3 - 4x + 4x^2 - x^3$ in the first quadrant is shaded. A solid of revolution is formed by rotating this region about the y -axis. Use the method of cylindrical shells to find the volume of this solid.

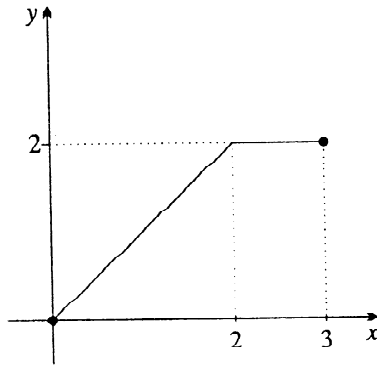
(b) Suppose the cubic equation $x^3 - 2x + 4 = 0$ has roots α, β and γ .

(i) Use the substitution $x^2 = y$ to show that a cubic equation which has roots α^2, β^2 and γ^2 is $y^3 - 4y^2 + 4y - 16 = 0$.

(ii) Factorise this new cubic into linear factors by initially grouping in pairs.

(iii) Hence show that the original equation has only one real root.

(c)



The graph of $y = f(x)$ is shown above. Sketch graphs of:

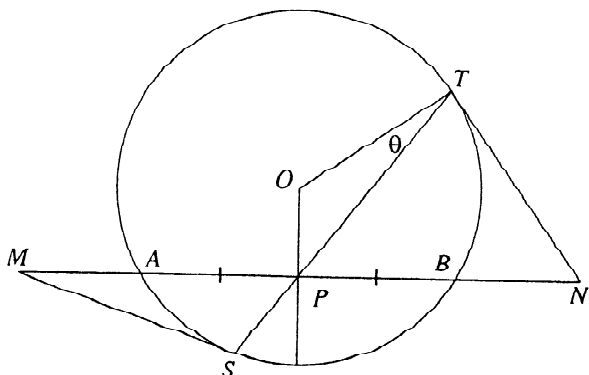
(i) $y = f(3 - x)$,

(ii) $y = f(|x|)$,

(iii) $y = \frac{1}{f(x)}$,

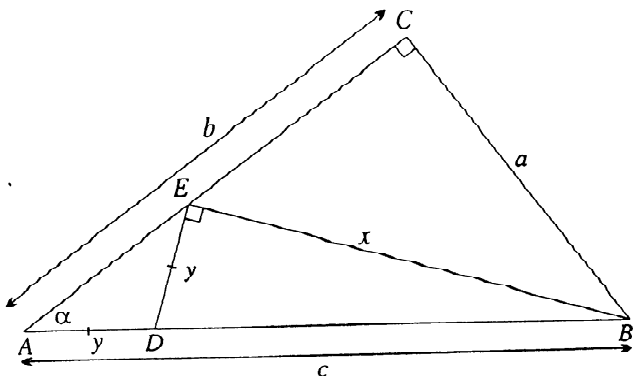
(iv) $y^2 = f(x)$.

4. (a)



In the diagram above, P is the midpoint of the chord AB in the circle with centre O . A second chord ST passes through P , and the tangents at the endpoints meet AB produced at M and N respectively.

- (i) Explain why $OPNT$ is a cyclic quadrilateral.
 - (ii) Explain why $OPSM$ is also cyclic.
 - (iii) Let $\angle OTS = \theta$. Show that $\angle ONP = \angle OMP = \theta$.
 - (iv) Hence prove that $AM = BN$.
- (b)



In the diagram above, $\triangle ABC$ is right-angled at C , with $a < b < c$. The points E on AC and D on AB are constructed so that $\angle BED$ is a right angle and $\triangle ADE$ is isosceles with $AD = DE$. Let $EB = x$, let $AD = DE = y$, and let $\angle CAB = \alpha$.

- (i) Prove that $\triangle ABC \parallel \triangle BEC$.
 - (ii) Show that $x = \frac{ca}{b}$.
 - (iii) Explain why $\angle BDE = 2\alpha$.
 - (iv) Hence show that $y = \frac{c(b^2 - a^2)}{2b^2}$.
- (c) Suppose the function $f(x)$ may be written as $f(x) = g(x) + h(x)$, where $g(x)$ is even and $h(x)$ is odd.
- (i) Find an expression for $g(x)$ in terms of $f(x)$ alone.
 - (ii) Hence write down $g(x)$ for the function $f(x) = e^x$.

5. (a) An object of mass m kg is projected vertically upwards from ground level

with an initial speed U m/s. Its characteristic shape results in air resistance which is proportional to the square of its velocity, that is, mkv^2 for some constant k . The only other force acting on the body is that due to gravity. Take upwards as the positive direction for displacement x . Take ground level as the origin of displacement.

(i) (α) Show that $\ddot{x} = -(g + kv^2)$.

(β) Use $\ddot{x} = \frac{dv}{dt}$ to show that the time T_u taken to reach the highest point of its flight is $T_u = \frac{1}{\sqrt{gk}} \tan^{-1} \left(U \sqrt{\frac{k}{g}} \right)$.

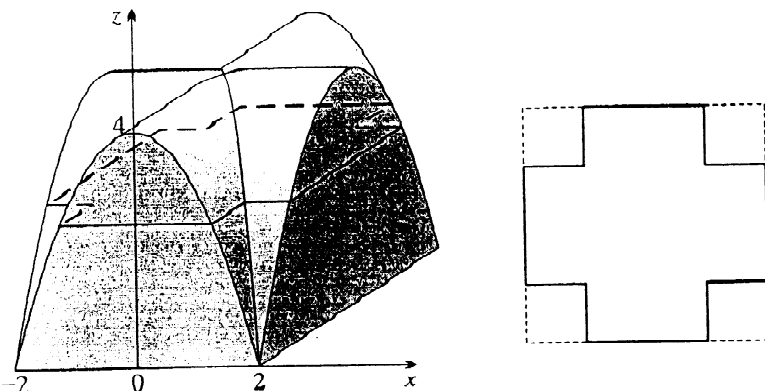
(ii) Let T_d be the time taken for the object to fall back down to the ground, and for convenience let $w = U \sqrt{\frac{k}{g}}$. It can be shown that $\sqrt{gk}(T_d - T_u)$ simplifies to the function $f(w) = \ln(w + \sqrt{w^2 + 1}) - \tan^{-1} w$.

(α) Evaluate $f(0)$.

(β) Determine $f'(w)$, and show that $f'(w) > 0$ for $w > 0$.

(γ) Hence show that it takes longer for the object to fall back to the ground than it does to reach its highest point.

(b)



A sandstone cap on the corner of a fence is shown above, formed in the shape of two intersecting parabolic cylinders. On the front face, the equation of the parabola is $z = 4 - x^2$, where x is the horizontal distance measured from the mid-point of the base of the front face, and z is the height. The shape of a horizontal slice of thickness dz taken at height z is also shown. It is a square with four smaller squares removed, one from each corner.

(i) Find x in terms of z .

(ii) Show that $V = \int_0^4 (4^2 - 4(2 - \sqrt{4 - z})^2) dz$.

(iii) Hence find the volume of stone in the cap.

6. (a) Let $P(z) = z^7 - 1$.

(i) Use de Moivre's theorem to find the roots of $P(z)$.

(ii) Hence write $P(z)$ as a product of:

(α) real and complex linear factors,

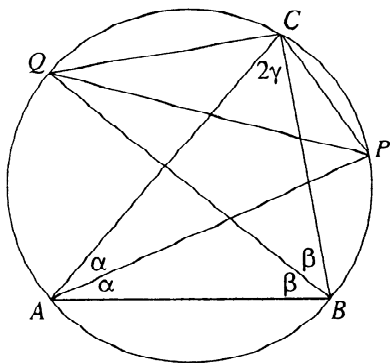
(β) real linear and irreducible quadratic factors.

(iii) Show that $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$.

- (iv) (α) Write down the quotient when $P(z)$ is divided by $z - 1$.
 (β) Hence show that $(1 - \cos \frac{2\pi}{7})(1 - \cos \frac{4\pi}{7})(1 - \cos \frac{6\pi}{7}) = \frac{7}{8}$.
 (b) For $n = 0, 1, 2, \dots$, the integral G_n is defined by $G_n = \int_0^\pi \frac{\sin nx}{3 - 2 \cos x} dx$.
 (i) Find G_0 and show that $G_1 = \frac{1}{2} \ln 5$.
 (ii) Show that $G_{n+1} + G_{n-1} - 3G_n = \frac{1}{n}((-1)^n - 1)$. [HINT: You may assume that $\sin A + \sin B = 2 \sin(\frac{A+B}{2}) \cos(\frac{A-B}{2})$.]
 (iii) Calculate G_3 .

7. (a) Let ω be a non-real cube root of unity.

- (i) Show that $1 + \omega + \omega^2 = 0$.
 (ii) Hence simplify $(1 + \omega)^2$.
 (iii) Show that $(1 + \omega)^3 = -1$.
 (iv) Use part (iii) to simplify $(1 + \omega)^{3n}$ and hence show that ${}^{3n}C_0 - \frac{1}{2}({}^{3n}C_1 + {}^{3n}C_2) + {}^{3n}C_3 - \frac{1}{2}({}^{3n}C_4 + {}^{3n}C_5) + {}^{3n}C_6 - \dots + {}^{3n}C_{3n} = (-1)^n$.
 [HINT: You may assume that $\Re(\omega) = -\frac{1}{2}$ and that $\Re(\omega^2) = -\frac{1}{2}$.]
 (b)



In the diagram above, AB is a fixed chord of a circle and C is a variable point on the major arc AB . The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the circle again at P and Q respectively. Let $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$ and $\angle BCA = 2\gamma$.

- (i) Show that $\angle PCQ = \alpha + \beta + 2\gamma$.
 (ii) Hence explain why the length of PQ is constant.
 (iii) Use the sine rule to show that $\frac{AB}{PQ} = 2 \sin \gamma$.
 (c) (i) Show that $\tan^2 \theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$.
 (ii) Hence show that $\tan^2(\frac{\alpha + \beta}{2}) - \tan \alpha \tan \beta = \frac{\cos(\alpha + \beta)(1 - \cos(\alpha - \beta))}{\cos \alpha \cos \beta (1 + \cos(\alpha + \beta))}$.
 (iii) Hence show that for $0 < \alpha < \frac{\pi}{4}$ and $0 < \beta < \frac{\pi}{4}$, $\sqrt{\tan \alpha \tan \beta} \leq \tan(\frac{\alpha + \beta}{2})$.

8. (a) Let $y = uv$ be the product of u and v , where u and v are functions of x .

- (i) Show that $y'' = uv'' + 2u'v' + u''v$.
 (ii) Develop similar expressions for y''' , y'''' and y''''' .
 (iii) Hence, or otherwise, find and simplify $\frac{d^5}{dx^5}((1 - x^2)e^{-x})$.
 (b) The *Bernstein polynomial* $B_{n,k}(t)$ of degree n and order k is defined by:

$$B_{n,k}(t) = {}^n C_k t^k (1 - t)^{n-k}, \text{ for } 0 \leq k \leq n.$$

(i) Write down the three Bernstein polynomials of degree 2, namely $B_{2,0}(t)$, $B_{2,1}(t)$ and $B_{2,2}(t)$.

(ii) The three fixed complex numbers α, β and γ are represented on the Argand diagram by the points A, B and C respectively. Three other complex numbers p, q and r are represented by the points P, Q and R respectively.

The point P divides the interval AB in the ratio $t : (1 - t)$.

The point Q also divides the interval BC in the ratio $t : (1 - t)$.

Likewise the point R divides the interval PQ in the ratio $t : (1 - t)$.

(α) Use the ratio division formula to find p and q in terms of α, β and γ .

(β) Hence show that $r = \alpha B_{2,0}(t) + \beta B_{2,1}(t) + \gamma B_{2,2}(t)$.

(γ) Given that $\alpha = 1 + i, \beta = 2 + 3i$ and $\gamma = 3 + i$, find the Cartesian equation of the locus of R as t varies.

(iii) (α) Show that $\sum_{k=0}^n B_{n,k}(t) = 1$.

(β) Show that for $r \leq k \leq n$, $\frac{{}^k C_r}{{}^n C_r} B_{n,k}(t) = {}^{n-r} C_{k-r} t^k (1 - t)^{n-k}$.

(γ) Using the previous two parts, or otherwise, show that $\sum_{k=2}^5 \frac{{}^k C_2}{{}^5 C_2} B_{5,k}(t) = t^2$.