Sydney Technical High School

Mathematics Extension 2

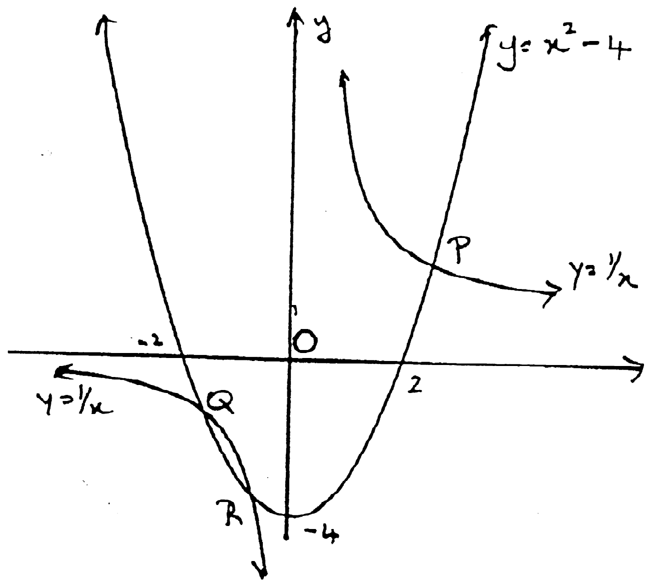
Trial HSC Examination 2001

1. **(a)** Find: **(i)** $\int \frac{dx}{x^2+2x+5}$ **(ii)** $\int_0^1 \frac{dx}{(x+1)\sqrt{x+1}}$
(b) Prove that $\sec x = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$ and hence find $\int \sec x \; dx$.
(c) **(i)** Find the exact value of $\int_0^1 xe^{-x} \; dx$ **(ii)** Find $\int \frac{5 \; dx}{(x+1)(x^2+4)}$
(d) Find $\int_k^1 \frac{dx}{x(x+1)}$ and hence prove that $\sum_{k=1}^{n} \int_k^1 \frac{dx}{x(x+1)} = \ln(n+1) - n\ln 2$.

2. **(a)** If $z = 3 - 4i$ find **(i)** $\overline{z}$ **(ii)** $|z|$ **(iii)** $\arg z$ **(iv)** $\arg(iz)$ **(v)** $\sqrt{z}$
(b) The complex number $z = x + iy$ is such that $|z - i| = \Im(z)$. Find, and describe geometrically, the locus of the point P representing z.
(c) Sketch the locus on the Argand diagram of the point Z representing the complex number z where $|z - 2i| = 1$. What is the least value of $\arg z$?
(d) A is the point representing the complex number $z = 2 + 3i$, while B represents the complex number iz. The point C is such that $AOBC$ is a square (where O is the origin). Find the co-ordinates of C.

3. **(a)** If one root of the polynomial equation $x^3 + ax^2 + bx + c = 0$ is the sum of the other two roots, show that $a^3 - 4ab + 8c = 0$.
(b) The polynomial $P(x) = x^3 + ax^2 + bx + 6$ where a and b are real numbers, has a zero of $1 - i$. Find a and b and express $P(x)$ as the product of two polynomials with real coefficients.

(c)



The curves $y = \frac{1}{x}$ and $y = x^2 - 4$ intersect at points P, Q, R as shown. P, Q and R have x-values α, β and γ. O is the origin.

(i) Show that α, β and γ are roots of $x^3 - 4x - 1 = 0$

(ii) Find a polynomial with numerical coefficients with roots α^2, β^2, and γ^2.

(iii) Find an expression for $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

(iv) Hence find the value of $OP^2 + OQ^2 + OR^2$.

4. (a) Given the hyperbola $9x^2 - 16y^2 = 144$ find

(i) the length of the major axis

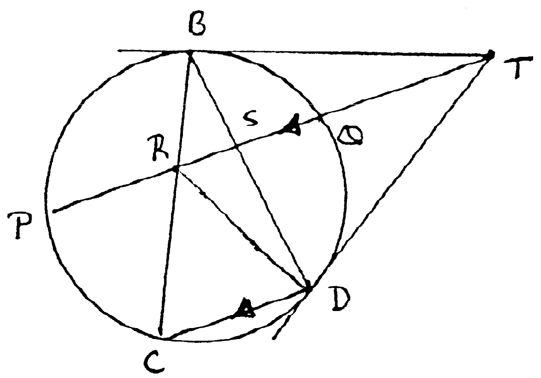
(ii) the eccentricity

(iii) the coordinates of the foci

(iv) the equations of the directrices

(v) the equations of the asymptotes

(b)



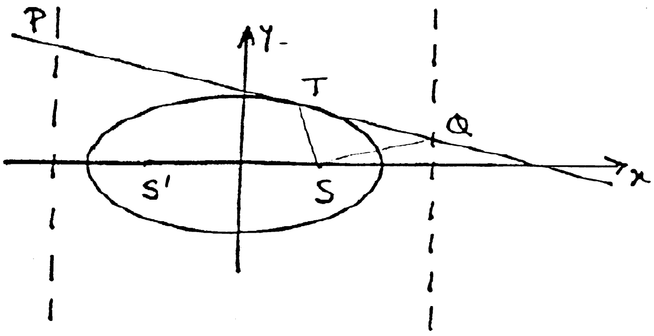
In the diagram, the chords PQ and CD are parallel. The tangent at D cuts the chord PQ at T. The other point of contact from T is B and BC cuts PQ at R.

(i) Copy the diagram.

(ii) Prove that $\angle BDT = \angle BRT$ and state why B, T, D and R are concyclic

(iii) Show that $\triangle RCD$ is isosceles.

(c)



The tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $T(a\cos\theta, b\sin\theta)$ meets the directrices of the ellipse at P and Q. S and S' are the foci. Show that $\angle TSQ = 90°$.

5. **(a)** Sketch, on separate axes, the following graphs, showing all important features (do not use calculus).

(i) $y = \sin^2 x$, $-2\pi \le x \le 2\pi$

(ii) $y = \ln(\frac{1}{x})$, $x > 0$

(iii) $y = \frac{\sin x}{x}$, $x > 0$

(iv) $y = \max(x, 1-x)$ where $\max(a,b) := \begin{cases} a & \text{for } a \ge b \\ b & \text{for } a < b \end{cases}$

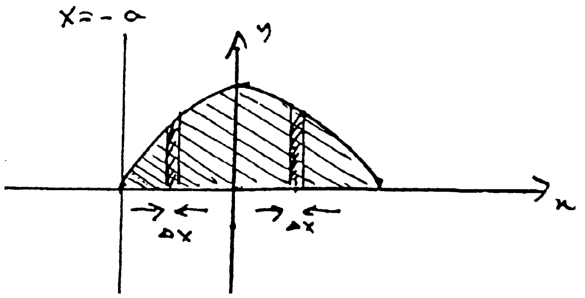
(b) **(i)** Use De Moivre's theorem to show that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ where n is an integer $(\cos\theta \ne 0)$

(ii) Use this result to show that the equation $(1+z)^4 + (1-z)^4 = 0$ has roots of $\pm i\tan\frac{\pi}{8}$, $\pm i\tan\frac{3\pi}{8}$

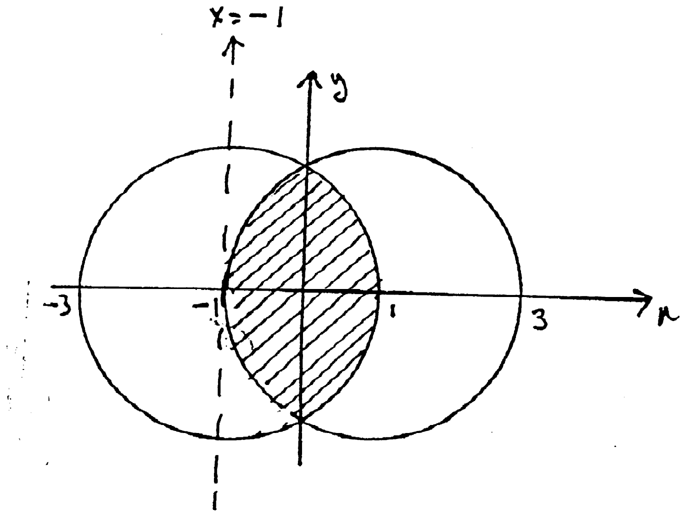
(iii) Hence, or otherwise, show that $\tan^2\frac{\pi}{8} = 3 - 2\sqrt{2}$.

6. **(a)** Find $\int_0^1 \sqrt{4-(1+x)^2}\,dx$

(b)



The curve $y = f(x)$ is reflected in the y-axis to give the shape shown. The strips shown both have width Δx and are equidistant from the y-axis.
(i) The shaded area is rotated around the line $x = -a$. Find each of the volumes of the two cylindrical shells as the two strips are rotated (Δx is small).
(ii) Show that the volume of the solid so formed is given by $V = 4\pi a \int_0^a f(x)\,dx$
(c)



Two circles, centres $(-1, 0)$ and $(1, 0)$ and of radii 2 units have a common region as shown, and this region is rotated about $x = -1$.
(i) Show that the volume of the solid formed is given by $V = 8\pi \int_0^1 \sqrt{4 - (x+1)^2}\,dx$
(ii) By using your answer to part **(a)** of this question above, find the exact volume of the solid.

7. (a) A particle moves in a straight line so that its distance from the origin at any time t is given by x and its velocity by v.
(i) The acceleration of the particle at a distance x is given by the equation $a = n^2(3 - x)$ where n is a constant. If the particle moves from rest from the origin ($x = 0$), show that $\frac{1}{2}v^2 - n^2(3x - \frac{1}{2}x^2) = 0$
(ii) Hence show that the particle never moves outside a certain interval and give

that interval.
(b) **(i)** Let $I_n = \int_1^e x(\ln x)^n \, dx$ where $n = 0, 1, 2, 3, \ldots$. Using integration by parts, show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$, $n = 1, 2, 3, \ldots$.
(ii) The area bounded by the curve $y = \sqrt{x}(\ln x)$, $x \geq 1$ the x-axis and the line $x = e$ is rotated about the x-axis through 2π radians. Find the exact value of the volume of the solid of revolution so formed.

8. **(a)** Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$ using the substitution $t = \tan \frac{x}{2}$
(b) A plane curve is defined by $x^2 + 2xy + y^5 = 4$. This curve has a horizontal tangent at the point $P(X, Y)$. By using implicit differentiation (or otherwise), show that X is the unique real root of $x^5 + x^2 + 4 = 0$.
(c) **(i)** If $x_1 > 1$ and $x_2 > 1$ show that $x_1 + x_2 > \sqrt{x_1 x_2}$
(ii) Use the principle of mathematical induction to show that, for $n \geq 2$, if $x_j > 1$ where $j = 1, 2, 3, \ldots, n$ then $\ln(x_1 + x_2 + \cdots + x_n) > \frac{1}{2^{n-1}}(\ln x_1 + \ln x_2 + \cdots + \ln x_n)$.