ABBOTSLEIGH

TRIAL HIGHER SCHOOL CERTIFICATE

1999

MATHEMATICS

4 UNIT

Time allowed: 3 hours

- All questions may be attempted
- Answer each question in a separate booklet
- All questions are of equal value
- Approved calculators may be used

Question One

Marks

(a) Evaluate $\int_0^1 \frac{2dx}{\sqrt{2-x^2}}$ — 2

(b) Find $\int xe^{-x}dx$ — 2

(c) Find $\int \sin^3 x \cos^2 x\,dx$ — 3

(d) Using the substitution $x=\frac{1}{u}$, where $u>0$ find $\int \frac{dx}{x\sqrt{x^2+1}}$ — 3

(e) Find $\int \sin(\log x)dx$ — 5

Question Two

Marks

(a) Let $z=\frac{-i}{1+i\sqrt{3}}$ — 2

(i) Sketch z on the Argand diagram.
(ii) Find the modulus and argument of z

(b) Let A = 1 + 2i and B = -3 + 4i — 3
Draw sketches to show the loci satisfied on the Argand diagram by

(i) $|z-A| = |B|$

(ii) $|z-A| = |z-B|$

(iii) $\arg(z-A)=\frac{\pi}{4}$

(c) (i) Solve the equation $z^4 = 1$ — 3

(ii) Hence find all solutions of the equation $z^4 = (z-1)^4$

(d) Use De Moivre's Theorem to express cos4θ in terms of cosθ — 3

(e) Express the roots of the equation $z^2+2(1+2i)z-(11+2i)=0$ in the form a + ib where a and b are real. — 2

(f) Draw Argand diagrams to represent the following regions — 2

(i) $1 \le |z+3-2i| \le 3$

(ii) $\frac{\pi}{6} \le \arg z \le \frac{\pi}{3}$

Question Three

(a) Make neat sketches of the following graphs, labelling any important features. — 4

(i) $y=\sin^2 2x$ for $-2\pi \le x \le 2\pi$

(ii) $|x|-|y|=1$

(b) (i) Express $\frac{3x+1}{(x+1)(x^2+1)}$ in the form $\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$ — 4

(ii) Hence find $\int \frac{3x+1}{(x+1)(x^2+1)}dx$

...on Three (continued) **Marks**

(c) Given $I_n = \int_0^{\pi/2} \cos^n x\,dx$ where n is a positive integer — 4

(i) Prove that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$

(ii) Hence evaluate I_4

(d) (i) Show that $1+i$ is a zero of the polynomial $P(x) = x^3 + x^2 - 4x + 6$ — 3

(ii) Express P(x) as a product of irreducible factors over the set of real numbers.

Question Four

(a) (i) Show that the tangent to the ellipse $\frac{x^2}{12} + \frac{y^2}{4} = 1$ at the point P(3, 1) has equation $x + y = 4$. — 8

(ii) If this tangent cuts the directrix in the fourth quadrant at the point T, and S is the corresponding focus, show that SP and ST are at right angles to each other.

(b) (i) Show that the tangent to the rectangular hyperbola $xy = c^2$ at the point $T(ct, \frac{c}{t})$ has equation $x + t^2y = 2ct$. — 7

(ii) The tangents to the rectangular hyperbola $xy = c^2$ at the points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$, where $pq = 1$, intersect at R.

Find the equation of the locus of R and state any restrictions on the values of x for this locus.

Question Five

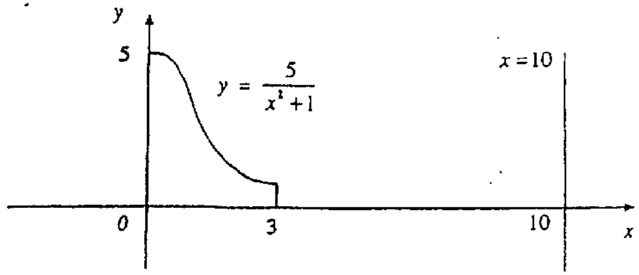
Marks

(a) (i) Sketch the curve $y = \sin^{-1} x$ — 5

(ii) Find the volume of the solid generated by rotating the region bounded by the curve $y = \sin^{-1} x$, the x-axis and the ordinate $x = 1$ about the y-axis. Use the method of slices.

(b) The base of a solid is the circle $x^2 + y^2 = 25$. Find the volume of the solid if every section perpendicular to the x-axis is a semi-circle whose diameter lies in the base of the solid. — 5

(c) — 5



The region bounded by the curve $y = \frac{5}{x^2+1}$, the x-axis and the lines $x = 0$ and $x = 3$ is rotated about the line $x = 10$.

(i) Use the method of cylindrical shells to show that the volume V cm^3 is given by $V = \int_0^3 \frac{100\pi - 10\pi x}{x^2+1}dx$

(ii) Hence find the volume V to the nearest cm^3.

Question 6

(a) The equation $x^4 + 4x^3 - 3x^2 - 4x - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$. 3

(b) Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ if it has a root of multiplicity 4. 6

(c) The chord of contact of the point $T(x_0, y_0)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the axes at M and N. If the mid-point of MN lies on the circle $x^2 + y^2 = 1$ what is the locus of T? 6

Question Seven

(a) A railway line has been constructed around a circular curve of radius 500 m. The distance between the rails is 1.5 m and the outside rail is 0.1 m above the inside rail. Find the speed that eliminates a sideways force on the wheels for a train on this curve. (Take g = 9.8 ms^{-2}.) 4

(b) A particle of mass m is set in motion with speed u. Subsequently the only force acting upon the particle directly opposes its motion and is of magnitude $mk(1 + v^2)$ where k is a constant and v is its speed at time t. 6

(i) Show that the particle is brought to rest after a time $\frac{1}{k}\tan^{-1} u$.

(ii) Find an expression for the distance travelled by the particle in this time.

(c) In Δ ABC, AB = AC. The bisector of $\angle$ ABC meets AC at M. The circle through A, B and M cuts BC at Q. Show, with reasons that AM = CQ. 5

Question Eight

Marks

(a) The tangent at a point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the x-axis at M, while the normal cuts the x-axis at N. Prove that OM.ON = a^2e^2. 5

(b) A particle is projected from the origin with initial velocity U to pass through a point (a,b). Prove that there are two possible trajectories if $(U^2 - gb)^2 > g^2(a^2 + b^2)$ 5

(c) A cone is placed with its vertex upward. A light string of length l metres is attached at one end to the vertex and the other end to a particle of mass m kg, which is made to describe a circle of uniform angular velocity ω in contact with the cone. Assume there is no friction on the cone's surface. Find the tension in the string, and the normal reaction of the surface. Hence, find the condition for this to happen. 5

End of paper