

4bbotsleigh university August 2002 (3 hours)

Total marks (120)  
Attempt Questions 1-8  
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find  $\int \frac{dx}{x^2 + 2x + 5}$ . Marks

2

(b) (i) Find real numbers  $A$ ,  $B$  and  $C$  such that

$$\frac{3x+7}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

2

(ii) Hence evaluate  $\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)} dx$

2

(c) Show  $\int e^{ax} \sin 3x dx = \frac{e^{ax}}{a^2+9} [a \sin 3x - 3 \cos 3x]$  and hence evaluate  $\int_0^{2\pi} e^{2x} \sin 3x dx$ .

4

(d)  $I_m = \int x^m e^x dx$

(i) Show that  $I_m = x^m e^x - m I_{m-1}$

2

(ii) Find the value of  $\int_1^2 x^2 e^x dx$

3

End of Question 1

QUESTION 2 (15 marks) Use a SEPARATE writing booklet.

(a) The complex number  $z$  is given by  $z = 1 + \frac{1+i}{1-i}$ . Find

(i)  $\bar{z}$ , giving your answer in the form  $x+iy$  where  $x$  and  $y$  are real.

2

(ii)  $iz$

1

(b) Find  $u$  and  $v$  if  $(u-iv)^2 = -21-20i$ .

4

(c) If  $z_1 = 1+3i$  and  $z_2 = 1-3i$ , find  $\left| \frac{z_1^{10}}{z_2^9} \right|$ .

3

(d) Shade the region in the complex plane for which  $2 \leq z + \bar{z} \leq 6$ .

2

(e) In the Argand diagram  $P(z)$  is a point in the first quadrant of the circle  $|z|=2$ . If  $\arg z = \theta$  find, in terms of  $\theta$ , expressions for

(i)  $\arg z^2$

1

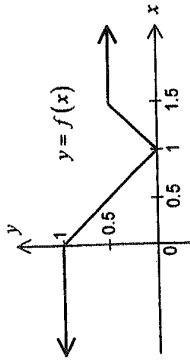
(ii)  $\arg(z-2)$

2

End of Question 2

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

- (a) The diagram below is a sketch of the function  $y = f(x)$ .



On separate diagrams sketch

- (i)  $y = f(-x)$
- (ii)  $y = \frac{1}{f(x)}$
- (iii)  $|y| = f(x)$

2

2

2

- (b) (i) Express  $\frac{x^2 - 8}{x^2 - 4}$  in the form  $c + \frac{d}{x^2 - 4}$  where  $c$  and  $d$  are integers.

1

- (ii) Draw a neat sketch of  $y = \frac{x^2 - 8}{x^2 - 4}$ . Clearly indicate the intercepts with the coordinate axes and the position and equation of all asymptotes.

4

- (iii) Shade the region  $R$  described by  $0 \leq y \leq \frac{x^2 - 8}{x^2 - 4}$  and  $3 \leq |x| \leq 4$ .

1

- (iv) Show that the area of  $R = 2 \left( 1 + \log_e \frac{3}{5} \right)$ .

3

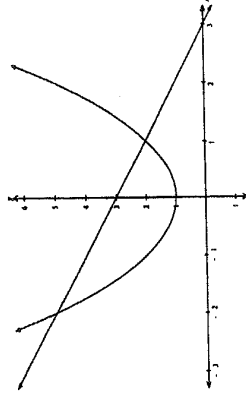
End of Question 3

QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

- (a) Show that  $2 - \sqrt{3}$  is a zero of the polynomial  $a(x) = x^3 - 15x + 4$ . Hence reduce  $a(x)$  to irreducible factors over the real field.
- (b) Factorise  $Q(x) = x^6 - 3x^2 + 2$  over the field of complex numbers, given that it has two double roots.
- (c) The area bounded by the curve  $y = x^2 + 1$  and the line  $y = 3 - x$  is rotated about the  $x$ -axis to form a solid.

3

4



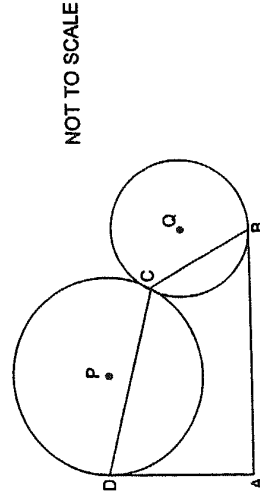
- (i) By considering slices perpendicular to the  $x$ -axis show that the area of one slice is given by  $A = \pi(8 - 6x - x^2 - x^4)$ .

2

- (ii) Hence find the volume of the solid formed.

2

- (d) Two circles touch each other at  $C$ .  $AD$  is a tangent to the circle with centre  $P$  and touches the circle at  $D$ .  $AB$  is a tangent to the circle with centre  $Q$  and touches the circle at  $B$ .



NOT TO SCALE

Copy the diagram into your answer booklet.

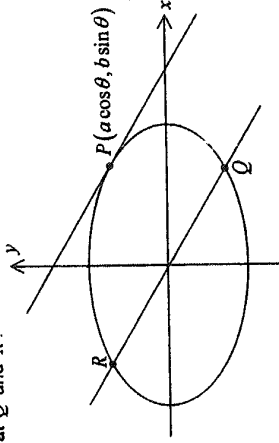
Prove that  $\angle BCD = 180^\circ - \frac{1}{2} \angle BAD$ .

4

End of Question 4

QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

- (a) Prove by mathematical induction  $3^n - 1 \geq 2n$  where  $n$  is a positive integer. 3
- (b)  $P(a \cos \theta, b \sin \theta)$  is a point on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with centre  $O$ . A line drawn through  $O$ , parallel to the tangent to the ellipse at  $P$ , meets the ellipse at  $Q$  and  $R$ .



- (i) Show that  $Q$  and  $R$  are the points  $(a \sin \theta, -b \cos \theta)$  and  $(-a \sin \theta, b \cos \theta)$ . 2
- (ii) Prove that the area of  $\Delta PQR$  is independent of the position of  $P$ . 3

- (c) A solid is built on the ellipse  $\frac{x^2}{9} + \frac{y^2}{4} = 1$  such that cross-sections perpendicular to the  $x$ -axis are squares with a side in the elliptical base. Find the volume of the solid. 4

- (d)  $Z$  and  $W$  represent the complex numbers  $z$  and  $w$  respectively. If  $|z| = 2$  and  $w = \frac{z+3}{z}$ , find the locus of  $W$ . 3

End of Question 6

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

- (a) The velocity  $v$  cm/s of a particle moving in a straight line is given by  $v^2 = 48 + 16x - 4x^2$  where  $x$  cm is the displacement of the particle from a fixed point. 3
- (i) Show that the particle is moving with simple harmonic motion. Hence write down the centre of motion. 2
- (ii) What is the amplitude of the motion? 2
- (iii) If initially the particle is at one of the extreme points, how far will it travel in the first  $\frac{3\pi}{4}$  seconds? 1
- (iv) Where is the particle when  $t = \frac{3\pi}{4}$  seconds? 1

- (b) A torus is generated by revolving the region  $x^2 + y^2 \leq 4$  about the line  $x = 5$ .

- (i) By using the method of cylindrical shells show that the volume of one shell is given by  $\Delta V = 4\pi(5-x)\sqrt{4-x^2}\Delta x$ . 2
- (ii) Hence find the volume of the torus. 2
- (c) Solve the equation  $4x^3 - 12x^2 + 11x - 3 = 0$  given that the roots form an arithmetic series. 3

End of Question 5

Marks

QUESTION 8 (15 marks) Use a SEPARATE writing booklet.

- (a) Show that  $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$ .  
Use this result to solve the equation  $8x^3 - 6x = -1$  for  $x$ . 5
- (b) If  $a > b$  and  $c > d$ , prove that  $ac + bd > ad + bc$ . 2
- (c) If  $a, b, c, d$  are positive numbers with  $a \geq c + d$  and  $b \geq c + d$ , prove that  $ab \geq ad + bc$ . 3
- (d) (i) Show that  $\frac{1 + \sin \theta + i \cos \theta}{1 + \sin \theta - i \cos \theta} = \sin \theta + i \cos \theta$  3  
(ii) Hence prove that  $\left( \frac{1 + \sin \theta + i \cos \theta}{1 + \sin \theta - i \cos \theta} \right)^n = \cos \left( \frac{n\pi}{2} - n\theta \right) + i \sin \left( \frac{n\pi}{2} - n\theta \right)$  2

where  $n$  is a positive integer.

END OF PAPER

Marks

QUESTION 7 (15 marks) Use a SEPARATE writing booklet.

- (a) The tangent to the hyperbola  $\frac{x^2}{16} - \frac{y^2}{9} = 1$  at  $P(4\sqrt{2}, 3)$  meets the asymptotes of the hyperbola at  $A$  and  $B$ .  
Show that  $P$  is the midpoint of  $AB$ . 4
- (b)  $LMN$  is an isosceles triangle with  $LM = LN$  and  $P$  is a point on the bisector of  $\angle MLN$ .  $MP$  produced meets  $LN$  at  $Q$  and  $NP$  produced meets  $LM$  at  $R$ .  
Prove that  $M, N, Q$  and  $R$  are concyclic. 4
- (c) (i) Show that the equation of the chord joining the points  $P\left(2p, \frac{2}{p}\right)$  and  $Q\left(2q, \frac{2}{q}\right)$  on the hyperbola  $xy = \frac{4}{x} + x + pqy = 2(p+q)$ . 3  
(ii) By letting  $q$  approach  $p$ , or otherwise, write down the equation of the tangent to the curve at  $P$ . 1  
(iii) If the chord in (i) passes through the point  $(2(p+q-1), 2)$ , and  $T$  is the point of intersection of the tangents at  $P$  and  $Q$ , show that the locus of  $T$  is  $y = x$ . 3

End of Question 7