Abbotsleigh ~ August 2002 (3 hours)

Total marks (120)
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{dx}{x^2+2x+5}$. **2**

(b) (i) Find real numbers A, B and C such that

$$\frac{3x+7}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$ **2**

(ii) Hence evaluate $\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx$ **2**

(c) Show $\int e^{ax} \sin 3x\, dx = \frac{e^{ax}}{a^2+9}\left[a\sin 3x - 3\cos 3x\right]$ and hence evaluate $\int_0^{2\pi} e^{2x} \sin 3x\, dx$. **4**

(d) $I_m = \int x^m e^x dx$

(i) Show that $I_m = x^m e^x - mI_{m-1}$ **2**

(ii) Find the value of $\int_1^2 x^2 e^x dx$ **3**

End of Question 1

Marks

QUESTION 2 (15 marks) Use a SEPARATE writing booklet.

(a) The complex number z is given by $z = 1 + \frac{1+i}{1-i}$. Find

(i) $\overline{z}$, giving your answer in the form $x+iy$ where x and y are real. **2**

(ii) iz **1**

(b) Find u and v if $(u-iv)^2 = -21-20i$. **4**

(c) If $z_1 = 1+3i$ and $z_2 = 1-3i$, find $\left|\frac{z_1^{10}}{z_2^9}\right|$. **3**

(d) Shade the region in the complex plane for which $2 \le z + \overline{z} \le 6$. **2**

(e) In the Argard diagram $P(z)$ is a point in the first quadrant of the circle $|z| = 2$. If $\arg z = \theta$ find, in terms of θ, expressions for

(i) $\arg z^2$ **1**

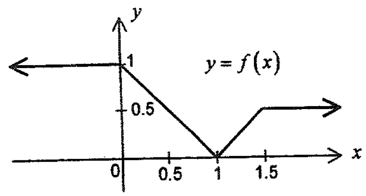
(ii) $\arg(z-2)$ **2**

End of Question 2

Marks

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below is a sketch of the function $y = f(x)$.



On separate diagrams sketch

(i) $y = f(-x)$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $|y| = f(x)$ **2**

(b) (i) Express $\dfrac{x^2-8}{x^2-4}$ in the form $c + \dfrac{d}{x^2-4}$ where c and d are integers. **1**

(ii) Draw a neat sketch of $y = \dfrac{x^2-8}{x^2-4}$. Clearly indicate the intercepts with the coordinate axes and the position and equation of all asymptotes. **4**

(iii) Shade the region R described by $0 \le y \le \dfrac{x^2-8}{x^2-4}$ and $3 \le |x| \le 4$. **1**

(iv) Show that the area of $R = 2\left(1 + \log_e \dfrac{3}{5}\right)$. **3**

End of Question 3

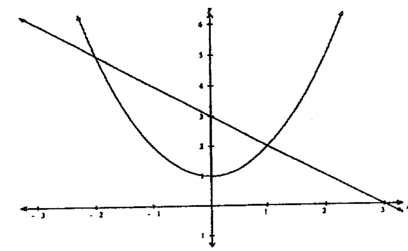
Marks

QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

(a) Show that $2-\sqrt{3}$ is a zero of the polynomial $a(x) = x^3 - 15x + 4$.
Hence reduce $a(x)$ to irreducible factors over the real field. **3**

(b) Factorise $Q(x) = x^6 - 3x^2 + 2$ over the field of complex numbers, given that it has two double roots. **4**

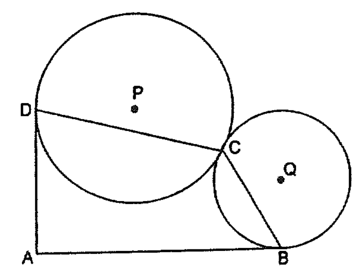
(c) The area bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x-axis to form a solid.



(i) By considering slices perpendicular to the x-axis show that the area of one slice is given by $A = \pi\left(8 - 6x - x^2 - x^4\right)$. **2**

(ii) Hence find the volume of the solid formed. **2**

(d) Two circles touch each other at C. AD is a tangent to the circle with centre P and touches the circle at D. AB is a tangent to the circle with centre Q and touches the circle at B.



NOT TO SCALE

Copy the diagram into your answer booklet.
Prove that $\angle BCD = 180^\circ - \dfrac{1}{2}\angle BAD$. **4**

End of Question 4

Marks

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

(a) The velocity v cm/s of a particle moving in a straight line is given by $v^2 = 48 + 16x - 4x^2$ where x cm is the displacement of the particle from a fixed point.

(i) Show that the particle is moving with simple harmonic motion. Hence write down the centre of motion. **3**

(ii) What is the amplitude of the motion? **2**

(iii) If initially the particle is at one of the extreme points, how far will it travel in the first $\frac{3\pi}{4}$ seconds? **2**

(iv) Where is the particle when $t = \frac{3\pi}{4}$ seconds? **1**

(b) A torus is generated by revolving the region $x^2 + y^2 \le 4$ about the line $x = 5$.

(i) By using the method of cylindrical shells show that the volume of one shell is given by $\Delta V = 4\pi(5-x)\sqrt{4-x^2}\,\Delta x$. **2**

(ii) Hence find the volume of the torus. **2**

(c) Solve the equation $4x^3 - 12x^2 + 11x - 3 = 0$ given that the roots form an arithmetic series. **3**

End of Question 5

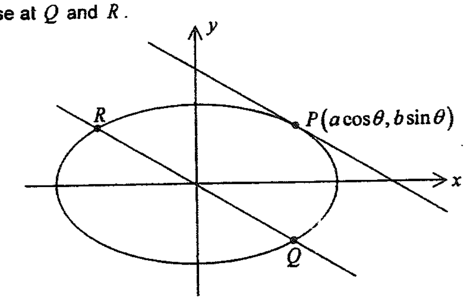
Marks

QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction $3^n - 1 \ge 2n$ where n is a positive integer. **3**

(b) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre O.

A line drawn through O, parallel to the tangent to the ellipse at P, meets the ellipse at Q and R.



(i) Show that Q and R are the points $(a\sin\theta, -b\cos\theta)$ and $(-a\sin\theta, b\cos\theta)$. **2**

(ii) Prove that the area of ΔPQR is independent of the position of P. **3**

(c) A solid is built on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ such that cross-sections perpendicular to the x-axis are squares with a side in the elliptical base.

Find the volume of the solid. **4**

(d) Z and W represent the complex numbers z and w respectively.

If $|z| = 2$ and $w = \frac{z+3}{z}$, find the locus of W. **3**

End of Question 6

Marks

QUESTION 7 (15 marks) Use a SEPARATE writing booklet.

(a) The tangent to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at $P\left(4\sqrt{2},3\right)$ meets the asymptotes of the hyperbola at A and B.

Show that P is the midpoint of AB. **4**

(b) LMN is an isosceles triangle with $LM = LN$ and P is a point on the bisector of $\angle MLN$. MP produced meets LN at Q and NP produced meets LM at R.

Prove that M, N, Q and R are concyclic. **4**

(c) (i) Show that the equation of the chord joining the points $P\left(2p,\frac{2}{p}\right)$ and $Q\left(2q,\frac{2}{q}\right)$ on the hyperbola $y=\frac{4}{x}$ is $x+pqy=2(p+q)$. **3**

(ii) By letting q approach p, or otherwise, write down the equation of the tangent to the curve at P. **1**

(iii) If the chord in (i) passes through the point $\left(2(p+q-1),2\right)$, and T is the point of intersection of the tangents at P and Q, show that the locus of T is $y=x$. **3**

End of Question 7

Marks

QUESTION 8 (15 marks) Use a SEPARATE writing booklet.

(a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Use this result to solve the equation $8x^3-6x=-1$ for x. **5**

(b) If $a>b$ and $c>d$, prove that $ac+bd>ad+bc$. **2**

(c) If a,b,c,d are positive numbers with $a \geq c+d$ and $b \geq c+d$, prove that $ab \geq ad+bc$. **3**

(d) (i) Show that $\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$ **3**

(ii) Hence prove that

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

where n is a positive integer. **2**

END OF PAPER