



ABBOTTSLEIGH

AUGUST 2003

YEAR 12

ASSESSMENT 4

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.

Total marks – 120

Attempt Questions 1-8

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks) Use a SEPARATE writing booklet.

(a) By completing the square, find $\int \frac{dx}{x^2 - 4x + 8}$ 2

(b) Use the substitution $x = \sin \theta$ to evaluate 3

$$\int_0^{\frac{\pi}{2}} \frac{x^2 dx}{\sqrt{1-x^2}}$$

(c) Use integration by parts to find $\int_1^e \frac{\ln x}{\sqrt{x}} dx$ 3

(d) (i) Find real numbers a , b and c such that 2

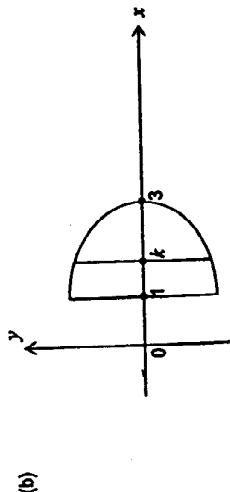
$$\frac{x+7}{(1+x^2)(1+x)} = \frac{ax+b}{1+x^2} + \frac{c}{1+x}$$

(ii) Find $\int \frac{x+7}{(1+x^2)(1+x)} dx$ 2

(e) Use the substitution $t = \tan \frac{x}{2}$ to find $\int \frac{\tan x}{1 + \cos x} dx$ 3

QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

- (a) The ellipse E has equation $\frac{x^2}{8} + \frac{y^2}{4} = 1$
- (i) Write down its eccentricity, the coordinates of its foci, S and S' , and the equation of each directrix. Sketch the ellipse E . 4
- (ii) If $P(x_1, y_1)$ is an arbitrary point on E , prove that the sum of the distances SP and $S'P$ is independent of the position of P . 2



The base of a particular solid is the region enclosed by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $x = 1$. Each cross-section of the solid perpendicular to the x -axis is an equilateral triangle.

- (i) Show that the area of the triangle at $x = k$ is $\frac{\sqrt{3}}{9}(36 - 4k^2)$ 2
- (ii) Find the volume of the solid. 3
- (iii) Consider a second solid which is obtained by rotating the region enclosed by the ellipse and the line $x = 1$ about the y -axis. Find the volume of the solid formed. 4

QUESTION 2 (15 marks) Use a SEPARATE writing booklet.

- (a) (i) Express $z = 1 + \sqrt{3}i$ in modulus-argument form. 2
- (ii) Show that $z^7 - 64z = 0$ 3
- (b) Let $z = x + iy$, where x and y are real numbers.
- (i) Solve $z\bar{z} + 2z = \frac{1}{4} + i$ 4
- (ii) Draw a neat sketch of the locus of $\operatorname{Re}(z) = |z - 2|$ 3
- (c) The points A, B, C, D on an Argand diagram represent the complex numbers a, b, c, d respectively. 3
- If $a + c = b + d$ and $a - c = i(b - d)$ find what type of quadrilateral is defined by $ABCD$. Clearly justify your answer.

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

- (a) Given the function $f(x) = x\sqrt{4 - x^2}$.
- (i) State its natural domain and show that it is an odd function. 2
- (ii) Show that on the curve $y = f(x)$, stationary points occur at $x = \pm\sqrt{2}$. Find the coordinates of the stationary points and determine their nature. 3
- (iii) Draw a neat sketch of the curve $y = f(x)$, indicating the above features, and given that there is a point of inflexion at the origin. 2
- (iv) On separate diagrams, sketch the curves
1. $y^2 = x^2(4 - x^2)$ 2
 2. $y = \frac{1}{f(x)}$ 2
- (b) Given that the sum of two of the roots of the equation $x^4 - x^3 - x^2 - x - 2 = 0$ is zero, find all four roots. 4

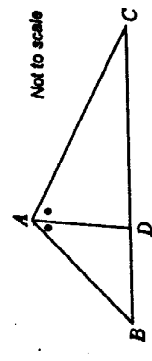
QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a) If α, β, γ are the roots of $2x^3 - 4x^2 - 3x - 1 = 0$, find the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$.

(b) Solve for x, y, z over the complex numbers:

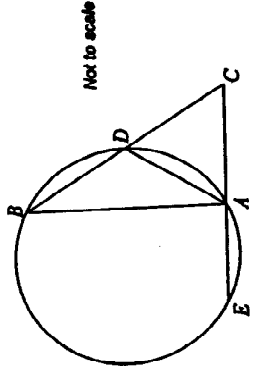
$$\begin{aligned} x + y + z &= 1 \\ xy + yz + zx &= 9 \\ xyz &= 9 \end{aligned}$$

(c) (i) In the triangle ABC , AD bisects angle BAC .



Prove that $\frac{BD}{DC} = \frac{BA}{AC}$

(ii)



In the diagram $AB = BC$ and AD bisects angle BAC .

Prove that $BD = CE$.

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

(a) Factorise $x^2 + 4x + 3$ and hence, or otherwise, show that the coefficient of x^4 in the expansion of $(x^2 + 4x + 3)^6$ is 61 695.

(b) (i) Prove that the equation of the tangent to the hyperbola $x^2 - y^2 = c^2$ at the point $P(x_1, y_1)$ is $xx_1 - yy_1 = c^2$.

(ii) This tangent meets the lines $y = x$ and $y = -x$ at Q and R respectively and O is the origin. Prove that the area of triangle OQR is constant.

(c) A particle moves in a straight line and its position x at any time t is given by $x = \sqrt{3} \cos 3t - \sin 3t$

(i) Show that the motion is simple harmonic.

(ii) Determine the period and amplitude of the motion.

Marks

QUESTION 7 (15 marks) Use a SEPARATE writing booklet.

(a) The minute hand OP and the hour hand OQ of a clock are 4cm and 3cm long respectively. Let $PQ = d$ be the distance between the tips of the hands of the clock.

(i) Show that $\frac{dPQ}{d\theta} = \frac{12 \sin \theta}{\sqrt{25 - 24 \cos \theta}}$ where θ is the acute angle between the hands of the clock. 2

(ii) Hence show that the rate of increase (in cm per hour) of the length of PQ at 9 o'clock is $\frac{22\pi}{5}$ cm/h. 3

(b) (i) If $f(x)$, $g(x)$ and $h(x)$ are distinct non-negative continuous functions of x in the interval $a \leq x \leq b$ and $f(x) < g(x) < h(x)$, explain why

$$\int_a^b f(x) dx < \int_a^b g(x) dx < \int_a^b h(x) dx$$

(ii) By considering the interval $0 < x < 1$ as an inequality, use algebra to show that

$$\frac{1}{2} x(1-x)^2 < \frac{x(1-x)}{1+x} < x(1-x)^2$$

(iii) Deduce that $\int_0^1 \frac{1}{2} x(1-x)^2 dx < \int_0^1 \frac{x(1-x)}{1+x} dx < \int_0^1 x(1-x)^2 dx$ 1

(iv) Given that $\int_0^1 \frac{x(1-x)}{1+x} dx = \frac{67}{12} - 8 \ln 2$, deduce that $\frac{83}{120} < \ln 2 < \frac{667}{960}$ 4

Marks

QUESTION 8 (15 marks) Use a SEPARATE writing booklet.

(a) A projectile is fired from the origin O with velocity V and angle of elevation α , where α is acute.

(i) By letting $g = \text{acceleration due to gravity}$ and $t = \frac{Y^2}{2g}$, derive the Cartesian equation of the parabolic path of the projectile. Show that as a quadratic equation in $\tan \alpha$, its Cartesian equation is

$$x^2 \tan^2 \alpha - 4kx \tan \alpha + (4ky + x^2) = 0$$

(ii) Show that the projectile can pass through the point (X, Y) in the first quadrant by firing at two different initial angles α_1 and α_2 if

$$X^2 < 4k^2 - 4kY$$

(iii) Let $\tan \alpha_1$ and $\tan \alpha_2$ be the two real roots of the quadratic equation in part (i). Show that $\tan \alpha_1 \tan \alpha_2 > 1$, and hence explain why it is impossible for both α_1 and α_2 to be less than 45° . 3

(b) It is given that $A > 0, B > 0$ and n is a positive integer.

(i) Divide $A^{n+1} - A^n B + B^{n+1} - B^n A$ by $A - B$ 2

(ii) Deduce that $A^{n+1} + B^{n+1} \geq A^n B + B^n A$ 1

(iii) Show by induction that $\left(\frac{A+B}{2}\right)^n \leq \frac{A^n + B^n}{2}$ 3

End of paper