Student Number___________________

ABBOTSLEIGH

AUGUST 2007
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

Outcomes assessed

HSC course

- **E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- **E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- **E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- **E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- **E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- **E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- **E7** uses the techniques of slicing and cylindrical shells to determine volumes
- **E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- **E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

From the Extension 1 Mathematics Course

Preliminary course

- **PE1** appreciates the role of mathematics in the solution of practical problems
- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives that require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks)
Use a SEPARATE writing booklet.

(a) Find $\int \sin^3\theta \, d\theta$. **2**

(b) (i) Express $\dfrac{3x+1}{(x+1)(x^2+1)}$ in the form $\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+1}$. **2**

(ii) Hence find $\int \dfrac{3x+1}{(x+1)(x^2+1)}$. **2**

(c) Use the substitution $x = 2\sin\theta$, or otherwise, to evaluate $\int_1^{\sqrt{3}} \dfrac{x^2}{\sqrt{4-x^2}}\,dx$. **3**

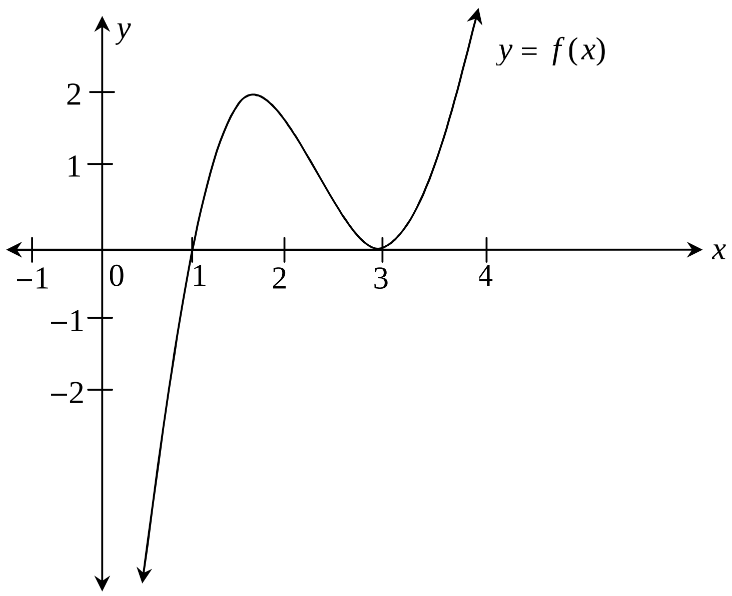
(d) Find $\int x^2\sqrt{3-x}\,dx$. **3**

(e) Evaluate $\int_0^1 \tan^{-1}\theta \, d\theta$. **3**

Marks

QUESTION 2 (15 marks)
Start a new writing booklet.

(a)



The diagram above is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = (f(x))^2$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \ln\left[f(x)\right]$ **2**

(iv) $y^2 = f(x)$ **2**

(b) (i) If $f'(x) = \dfrac{2-x}{x^2}$ and $f(1) = 0$, find $f''(x)$ and $f(x)$. **3**

(ii) Explain why the graph of $f(x)$ has only one turning point and find the value of the function at that point, stating whether it is a maximum or a minimum value. **2**

(iii) Show that $f(4)$ and $f(5)$ have opposite signs and draw a sketch of $f(x)$. **2**

Marks

QUESTION 3 (15 marks)
Start a new writing booklet.

(a) Express $(\sqrt{3}+i)^8$ in the form $x+iy$. **3**

(b) On an Argand diagram, sketch the region where the inequalities **3**

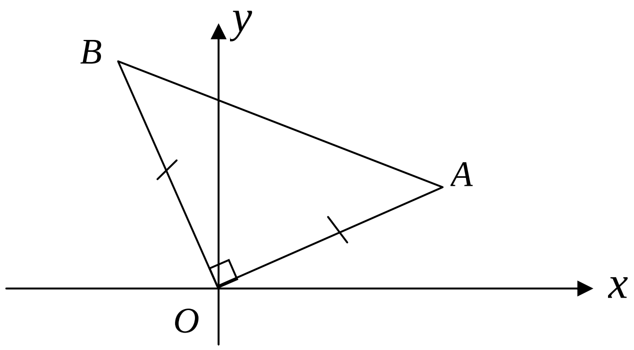
$$|z| \le 3 \quad \text{and} \quad -\frac{2\pi}{3} \le \arg\,(z+2) \le \frac{\pi}{6} \quad \text{both hold.}$$

(c) Show that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$. **3**

(d) (i) Express $z=\dfrac{-1+i}{\sqrt{3}+i}$ in modulus-argument form. **2**

(ii) Hence evaluate $\cos\dfrac{7\pi}{12}$ in surd form. **2**

(e) The Argand diagram below shows the points A and B which represent the complex numbers z_1 and z_2 respectively.



Given that ΔBOA is a right-angled isosceles triangle, show that $(z_1+z_2)^2=2z_1z_2$. **2**

Marks

QUESTION 4 (15 marks)
Start a new writing booklet.

(a) If $z = 1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$ where p and q are real, find p and q. **3**

(b) Show that if the polynomial $f(x) = x^3 + px + q$ has a multiple root, then $4p^3 + 27q^2 = 0$. **3**

(c) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$, the x-axis and the line $x = \dfrac{\pi}{2}$. **3**

Find the volume of the solid if every cross-section perpendicular to the base and the $x-$axis is a square.

(d) (i) Find the five roots of the equation $z^5 = 1$. Give the roots in modulus-argument form. **2**

(ii) Show that $z^5 - 1$ can be factorised in the form :

$$z^5 - 1 = (z - 1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$$ **2**

(iii) Hence show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$. **2**

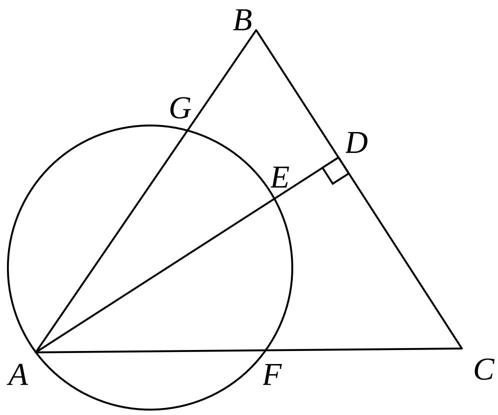
Marks

QUESTION 5 (15 marks)
Start a new writing booklet.

(a) The ellipse $(x-1)^2+\dfrac{y^2}{4}=1$ is rotated about the $y-$axis.

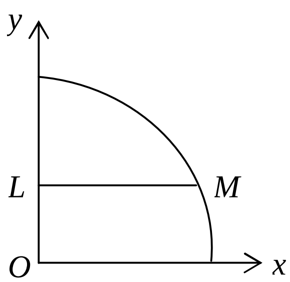
Use the method of slicing to find the volume of the solid formed by the rotation. **4**

(b) In the triangle ABC, AD is the perpendicular from A to BC. E is any point on AD and the circle drawn with AE as diameter cuts AC at F and AB at G. **4**



Prove B, G, F and C are concyclic.

(c) The diagram below shows the part of the circle $x^2+y^2=a^2$ in the first quadrant.



(i) If the horizontal line LM through $L(0,b)$, where $0<b<a$, divides the area between the curve and the coordinates axes into two equal parts, show that

$$\sin^{-1}\frac{b}{a}+\frac{b\sqrt{a^2-b^2}}{a^2}=\frac{\pi}{4}.$$ **3**

(ii) If the radius of the circle is 1 unit, show that b can be found by solving the equation

$\sin 2\theta=\dfrac{\pi}{2}-2\theta$, where $\theta=\sin^{-1}b$. **3**

(iii) Without attempting to solve the equation, how could θ (and hence b) be approximated? **1**

Marks

QUESTION 6 (15 marks)
Start a new writing booklet.

(a) An ellipse has equation $\frac{x^2}{4}+\frac{y^2}{3}=1$ with vertices $A(2,0)$ and $A'(-2,0)$. P is a point (x_1, y_1) on the ellipse.

(i) Find its eccentricity, coordinates of its foci, S and S', and the equations of its directrices. **3**

(ii) Prove that the sum of the distances SP and $S'P$ is independent of the position of P. **2**

(iii) Show that the equation of the tangent to the ellipse at P is $\frac{xx_1}{4}+\frac{yy_1}{3}=1$. **2**

(iv) The tangent at $P(x_1, y_1)$ meets the directrix at T. Prove that angle PST is a right angle. **3**

(b) If $a+b+c=1$,

(i) Prove $a^2+b^2 \ge 2ab$. **1**

(ii) Prove $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 9$. **2**

(iii) Prove $(1-a)(1-b)(1-c) \ge 8abc$. **2**

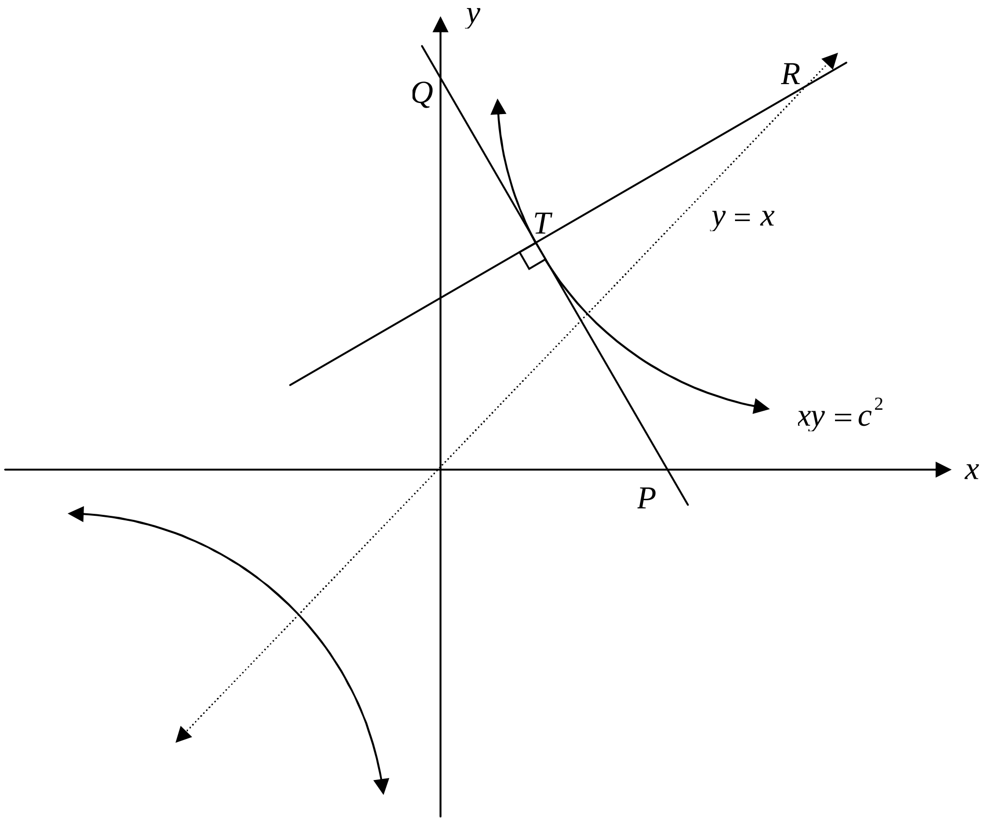
Marks

QUESTION 7 (15 marks)
Start a new writing booklet.

(a) The point $T(ct, \frac{c}{t})$ lies on the hyperbola $xy = c^2$.

The tangent at T meets the x – axis at P and the y – axis at Q.

The normal at T meets the line $y = x$ at R.



not to scale

You may assume that the tangent at T has equation $x + t^2 y = 2ct$.

(i) Find the coordinates of P and Q. **2**

(ii) Find the equation of the normal at T. **2**

(iii) Show that the x – coordinate of R is $x = \frac{c}{t}(t^2 + 1)$. **2**

(iv) Prove that ΔPQR is isosceles. **3**

(b) (i) If $I_n = \int \frac{dx}{\left(x^2+1\right)^n}$ prove that $I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right]$. **4**

(ii) Hence evaluate $\int_0^1 \frac{dx}{\left(x^2+1\right)^2}$. **2**

QUESTION 8 (15 marks) **Marks**
Start a new writing booklet.

(a) A plane of mass M kg on landing, experiences a variable resistive force due to air resistance of magnitude Bv^2 newtons, where v is the speed of the plane. That is, $M\ddot{x} = -Bv^2$.

(i) Show that the distance (D_1) travelled in slowing the plane from speed V to speed U under the effect of air resistance only, is given by: **4**

$$D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$$

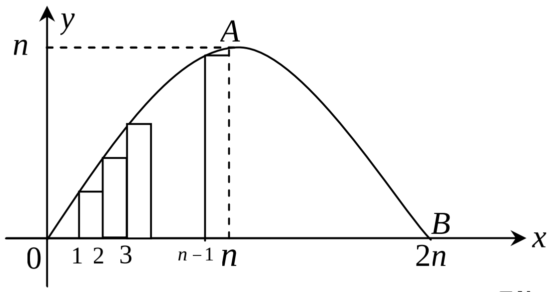
After the brakes are applied, the plane experiences a constant resistive force of A Newtons (due to brakes) as well as a variable resistive force, Bv^2. That is, $M\ddot{x} = -(A + Bv^2)$.

(ii) After the brakes are applied when the plane is travelling at speed U, show that the distance D_2 required to come to rest is given by: **4**

$$D_2 = \frac{M}{2B}\ln\left[1 + \frac{B}{A}U^2\right].$$

(iii) Use the above information to estimate the total stopping distance after landing, for a 100 tonne plane if it slows from 90 $\mathrm{m/s^2}$ to 60 $\mathrm{m/s^2}$ under a resistive force of $125v^2$ Newtons and is finally brought to rest with the assistance of a constant braking force of magnitude 75 000 Newtons. **2**

(b)



The diagram above represents the curve $y = n\sin\dfrac{\pi x}{2n},\ 0 \le x \le 2n,$ where n is any integer $n \ge 2$.

The points $O(0,0)$, $A(n,n)$ and $B(2n,0)$ lie on this curve.

(i) By considering the areas of the lower rectangles of width 1 from $x = 0$ to $x = n$, prove that

$$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \ldots. + \sin\frac{\pi(n-1)}{2n} < \frac{2n}{\pi}.$$ **3**

(ii) Hence or otherwise, explain why $2n\sum_{r=1}^{n-1}\sin\dfrac{\pi r}{2n} < \dfrac{\pi n^2}{2}$. **2**

END OF PAPER