Student Number___________________

ABBOTSLEIGH

AUGUST 2007
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Outcomes assessed

HSC course

E1 appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
E5 uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
E7 uses the techniques of slicing and cylindrical shells to determine volumes
E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
E9 communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

From the Extension 1 Mathematics Course

Preliminary course

PE1 appreciates the role of mathematics in the solution of practical problems
PE2 uses multi-step deductive reasoning in a variety of contexts
PE3 solves problems involving inequalities, polynomials, circle geometry and parametric representations
PE4 uses the parametric representation together with differentiation to identify geometric properties of parabolas
PE5 determines derivatives that require the application of more than one rule of differentiation
PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics
HE2 uses inductive reasoning in the construction of proofs
HE3 uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
HE4 uses the relationship between functions, inverse functions and their derivatives
HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
HE6 determines integrals by reduction to a standard form through a given substitution
HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks)
Use a SEPARATE writing booklet.

(a) Find $\int \sin^3\theta \, d\theta$. **2**

(b) (i) Express $\dfrac{3x+1}{(x+1)(x^2+1)}$ in the form $\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+1}$. **2**

(ii) Hence find $\int \dfrac{3x+1}{(x+1)(x^2+1)}$. **2**

(c) Use the substitution $x = 2\sin\theta$, or otherwise, to evaluate $\int_1^{\sqrt{3}} \dfrac{x^2}{\sqrt{4-x^2}}\,dx$. **3**

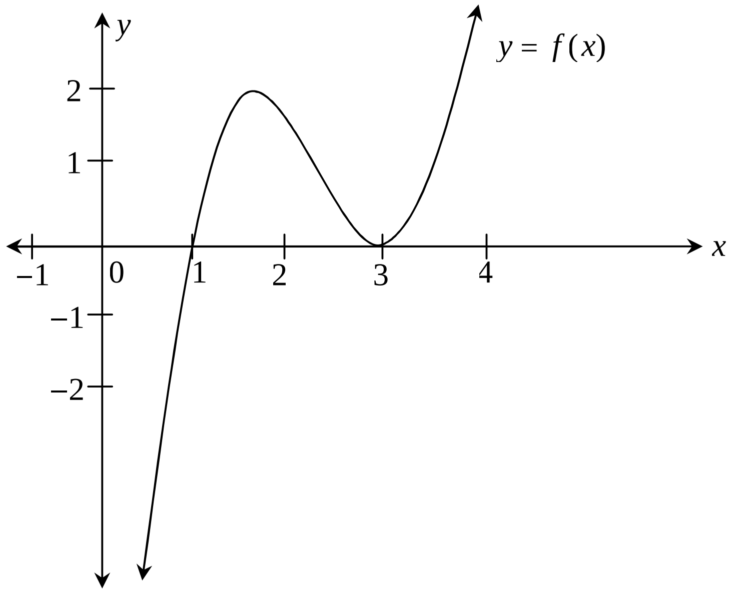
(d) Find $\int x^2\sqrt{3-x}\,dx$. **3**

(e) Evaluate $\int_0^1 \tan^{-1}\theta \; d\theta$. **3**

Marks

QUESTION 2 (15 marks)
Start a new writing booklet.

(a)



The diagram above is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = (f(x))^2$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \ln\left[f(x)\right]$ **2**

(iv) $y^2 = f(x)$ **2**

(b) (i) If $f'(x) = \dfrac{2-x}{x^2}$ and $f(1) = 0$, find $f''(x)$ and $f(x)$. **3**

(ii) Explain why the graph of $f(x)$ has only one turning point and find the value of the function at that point, stating whether it is a maximum or a minimum value. **2**

(iii) Show that $f(4)$ and $f(5)$ have opposite signs and draw a sketch of $f(x)$. **2**

Marks

QUESTION 3 (15 marks)
Start a new writing booklet.

(a) Express $(\sqrt{3}+i)^8$ in the form $x+iy$. **3**

(b) On an Argand diagram, sketch the region where the inequalities **3**

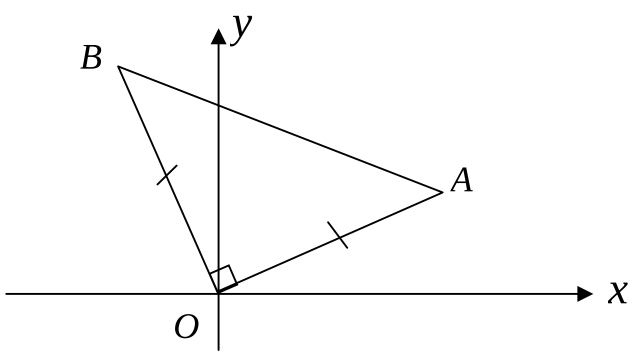
$$|z| \le 3 \text{ and } -\frac{2\pi}{3} \le \arg(z+2) \le \frac{\pi}{6} \text{ both hold.}$$

(c) Show that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$. **3**

(d) (i) Express $z=\dfrac{-1+i}{\sqrt{3}+i}$ in modulus-argument form. **2**

(ii) Hence evaluate $\cos\dfrac{7\pi}{12}$ in surd form. **2**

(e) The Argand diagram below shows the points A and B which represent the complex numbers z_1 and z_2 respectively.



Given that ΔBOA is a right-angled isosceles triangle, show that $(z_1+z_2)^2=2z_1z_2$. **2**

Marks

QUESTION 4 (15 marks)
Start a new writing booklet.

(a) If $z = 1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$ where p and q are real, find p and q. **3**

(b) Show that if the polynomial $f(x) = x^3 + px + q$ has a multiple root, then $4p^3 + 27q^2 = 0$. **3**

(c) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$, the x-axis and the line $x = \frac{\pi}{2}$. **3**

Find the volume of the solid if every cross-section perpendicular to the base and the $x-$axis is a square.

(d) (i) Find the five roots of the equation $z^5 = 1$. Give the roots in modulus-argument form. **2**

(ii) Show that $z^5 - 1$ can be factorised in the form :

$$z^5 - 1 = (z-1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$$ **2**

(iii) Hence show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **2**

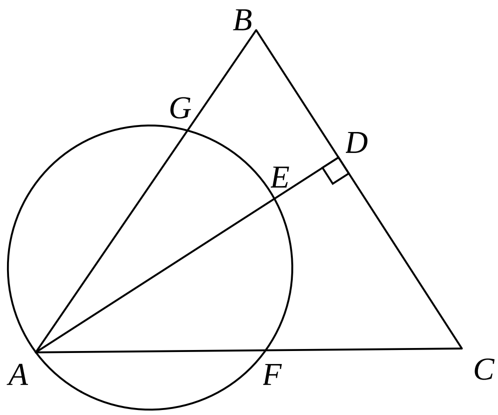
Marks

QUESTION 5 (15 marks)
Start a new writing booklet.

(a) The ellipse $(x-1)^2+\dfrac{y^2}{4}=1$ is rotated about the $y-$axis.

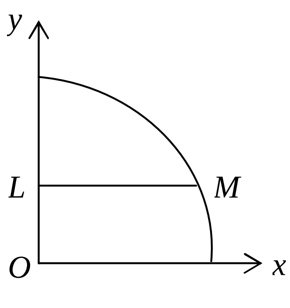
Use the method of slicing to find the volume of the solid formed by the rotation. **4**

(b) In the triangle ABC, AD is the perpendicular from A to BC. E is any point on AD and the circle drawn with AE as diameter cuts AC at F and AB at G. **4**



Prove B, G, F and C are concyclic.

(c) The diagram below shows the part of the circle $x^2+y^2=a^2$ in the first quadrant.



(i) If the horizontal line LM through $L(0,b)$, where $0<b<a$, divides the area between the curve and the coordinates axes into two equal parts, show that

$$\sin^{-1}\frac{b}{a}+\frac{b\sqrt{a^2-b^2}}{a^2}=\frac{\pi}{4}.$$ **3**

(ii) If the radius of the circle is 1 unit, show that b can be found by solving the equation

$\sin 2\theta=\dfrac{\pi}{2}-2\theta$, where $\theta=\sin^{-1}b$. **3**

(iii) Without attempting to solve the equation, how could θ (and hence b) be approximated? **1**

Marks

QUESTION 6 (15 marks)
Start a new writing booklet.

(a) An ellipse has equation $\frac{x^2}{4}+\frac{y^2}{3}=1$ with vertices $A(2,0)$ and $A'(-2,0)$. P is a point (x_1, y_1) on the ellipse.

(i) Find its eccentricity, coordinates of its foci, S and S', and the equations of its directrices. **3**

(ii) Prove that the sum of the distances SP and $S'P$ is independent of the position of P. **2**

(iii) Show that the equation of the tangent to the ellipse at P is $\frac{xx_1}{4}+\frac{yy_1}{3}=1$. **2**

(iv) The tangent at $P(x_1, y_1)$ meets the directrix at T. Prove that angle PST is a right angle. **3**

(b) If $a+b+c=1$,

(i) Prove $a^2+b^2 \geq 2ab$. **1**

(ii) Prove $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$. **2**

(iii) Prove $(1-a)(1-b)(1-c) \geq 8abc$. **2**

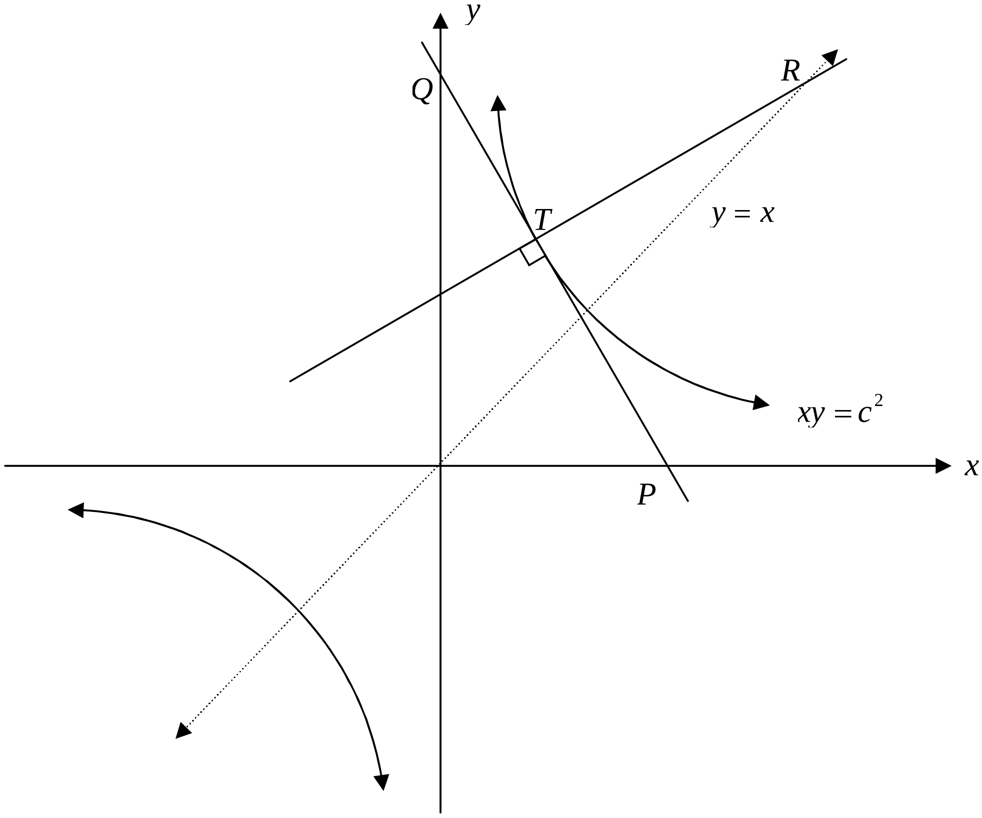
Marks

QUESTION 7 (15 marks)
Start a new writing booklet.

(a) The point $T(ct, \frac{c}{t})$ lies on the hyperbola $xy = c^2$.

The tangent at T meets the x – axis at P and the y – axis at Q.

The normal at T meets the line $y = x$ at R.



not to scale

You may assume that the tangent at T has equation $x + t^2 y = 2ct$.

(i) Find the coordinates of P and Q. **2**

(ii) Find the equation of the normal at T. **2**

(iii) Show that the x – coordinate of R is $x = \frac{c}{t}(t^2 + 1)$. **2**

(iv) Prove that ΔPQR is isosceles. **3**

(b) (i) If $I_n = \int \frac{dx}{\left(x^2+1\right)^n}$ prove that $I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right]$. **4**

(ii) Hence evaluate $\int_0^1 \frac{dx}{\left(x^2+1\right)^2}$. **2**

QUESTION 8 (15 marks) **Marks**
Start a new writing booklet.

(a) A plane of mass M kg on landing, experiences a variable resistive force due to air resistance of magnitude Bv^2 newtons, where v is the speed of the plane. That is, $M\ddot{x} = -Bv^2$.

(i) Show that the distance (D_1) travelled in slowing the plane from speed V to speed U under the effect of air resistance only, is given by: **4**

$$D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$$

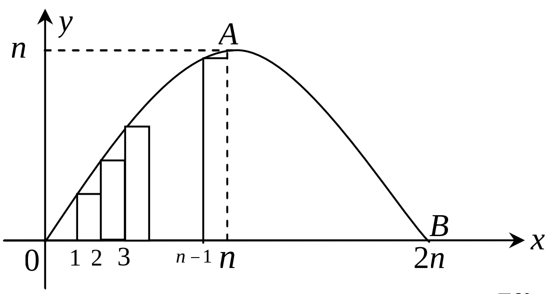
After the brakes are applied, the plane experiences a constant resistive force of A Newtons (due to brakes) as well as a variable resistive force, Bv^2. That is, $M\ddot{x} = -(A + Bv^2)$.

(ii) After the brakes are applied when the plane is travelling at speed U, show that the distance D_2 required to come to rest is given by: **4**

$$D_2 = \frac{M}{2B}\ln\left[1 + \frac{B}{A}U^2\right].$$

(iii) Use the above information to estimate the total stopping distance after landing, for a 100 tonne plane if it slows from 90 $\mathrm{m/s^2}$ to 60 $\mathrm{m/s^2}$ under a resistive force of $125v^2$ Newtons and is finally brought to rest with the assistance of a constant braking force of magnitude 75 000 Newtons. **2**

(b)



The diagram above represents the curve $y = n\sin\frac{\pi x}{2n},\ 0 \le x \le 2n,$ where n is any integer $n \ge 2$.

The points $O(0,0)$, $A(n,n)$ and $B(2n,0)$ lie on this curve.

(i) By considering the areas of the lower rectangles of width 1 from $x = 0$ to $x = n$, prove that **3**

$$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + ... + \sin\frac{\pi(n-1)}{2n} < \frac{2n}{\pi}.$$

(ii) Hence or otherwise, explain why $2n\sum_{r=1}^{n-1}\sin\frac{\pi r}{2n} < \frac{\pi n^2}{2}$. **2**

END OF PAPER