Ascham School
Trial Higher School Certificate
Mathematics 4 unit

July 1999 **Time allowed: 3 hou**

Instructions to Students

1. Attempt all questions
2. All questions are of equal value
3. Answer each question in a separate booklet
4. Marks may not be awarded for careless of badly arranged work
5. Approved calculators may be used
6. Table of Standard Integrals are provided

Question 1 (15 marks)

a) Find $\int 7x\sqrt{4x^2-3}dx$ 2

Evaluate the following definite integrals

(b) (i) $\int_0^{\sqrt{2}} \sqrt{4-x^2}dx$ 3

(ii) $\int_0^{\pi} x\sin x dx$ 3

(iii) $\int_2^4 \frac{dx}{x^2-4x+8}$ 3

(iv) $\int_{-1}^{1} \frac{4+x^2}{4-x^2}dx$ 4

Question 2 (15 marks) START A NEW BOOKLET

a) (i) Solve $x^2-3ix+4=0$ 2

(ii) Express $\sqrt{12-5i}$ in the form $a+ib$, where a,b are real 4

(iii) Find the locus of z, where $z=\frac{u-i}{u-2}$ 5

α) If u is purely real

β) If u moves around a unit circle

(iv) Indicate on an Argand diagram the region in which both the following inequalities are satisfied. 4

$$|z-(3+i)|\le 3 \text{ and } \frac{\pi}{4}\le \arg[z-(1+i)]\le\frac{\pi}{2}$$

Question 3 (15 marks) START A NEW BOOKLET

a) Let $I_n=\int_0^{\frac{\pi}{4}} \tan^n x dx$ where n is an integer and $n\ge 3$ 4

Show that $I_n+I_{n-2}=\frac{1}{n-1}$ and hence evaluate I_5

b) (i) If $u=\frac{1+i}{\sqrt{2}}$, show that $u^4=-1$

(ii) On an Argand diagram illustrate the roots of the equation $z^4=1$ 11

(iii) On the same diagram illustrate the roots of the equation $z^4=-1$

(iv) Hence or otherwise write down the solutions of the equation $z^8-1=0$

Question 4 (15 marks) START A NEW BOOKLET

a) The roots of the polynomial $P(x) = 4x^3 - 12x^2 + 11x - 3$ are in arithmetic sequence Solve P(x) = 0 over the real number system. 4

b) (i) Prove that if Q(x) is a polynomial with a real root at x = a of multiplicity r+1 then Q'(x) has r – fold roots at x = a. 7

(ii) Solve the equation $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$ given that it has a root of multiplicity 2 over C.

c) If $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ 4

(ii) Hence by dividing throughout by z^2 or otherwise, solve the equation $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$, given that $|z| = 1$. 2

Question 5 (15marks) START A NEW BOOKLET

a) (i) Show that the equation of the tangent and normal at $P(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ and $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ respectively. 3

(ii) The tangent and normal at P cut the y – axis at A and B respectively. Find the coordinates of A and B. 2

(iii) Show that the focus S lies on the circumference of the semi circle which has diameter AB. 3

b) (i) Determine the real values of k for which $\frac{x^2}{4+k} + \frac{y^2}{9+k} = 1$ defines 3

α) an ellipse
β) an hyperbola

(ii) If k = -5 in the above equation, find the eccentricity, the coordinates of the foci and the equations of the directrices of the conic. 2

(iii) Draw a neat sketch of the conic indicating all key features. 2

Question 6 (15 marks) START A NEW BOOKLET

a) Let $f(x) = \frac{4}{x} - x$. Provide separate half page sketches of the graphs of the following:

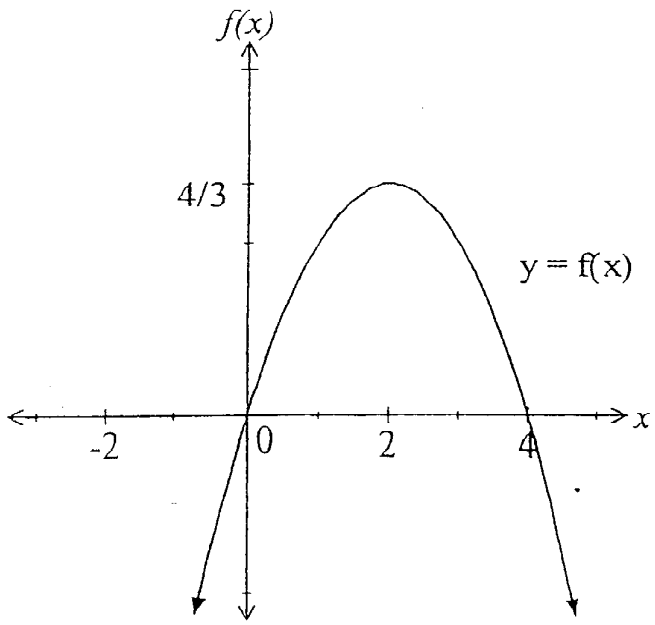
(i) $y = f(x)$ 2

(ii) $y = \sqrt{f(x)}$ 2

(iii) $y = e^{f(x)}$ 2

Label each graph carefully

b)



(i) Use the diagram to find the values of a, b, c *given* $f(x) = ax^2 + bx + c$ 2

(ii) Solve $-1 \le f(x) \le 1$ 3

(iii) Hence or otherwise sketch 4

α) $y = \ln[f(x)]$

β) $y = \cos^{-1}[f(x)]$

Question 7 (15 marks) START A NEW BOOKLET

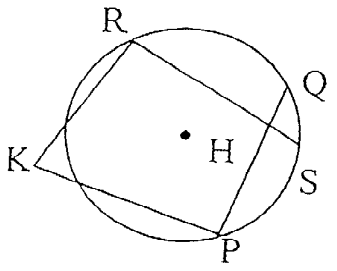
a) The base of a solid is a circular region of radius a units. Find the volume every cross section of a plane perpendicular to a certain diameter is a squar with one side lying in the base.

b) Find, by the method of cylindrical shells, the volume of the solid generate when the region bounded by the curve $y = x^2 + 1$, the line $x = 2$ and th coordinate axes is rotated about the line $x = 3$. 5

c) Find the value of x such that $\sin x = \cos 5x$ and $0 < x < \pi$ 6

Question 8 (15 marks) START A NEW BOOKLET

a) PQ and RS are 2 chords of a circle. PQ and RS intersect at H. K is a poir such that angle KPQ and angle KRS are right angles. Show that K produced is perpendicular to QS.



(b) A parachutist of mass m falls to ground from a plane. Given that air resistance is proportional to the square of his speed v: 6 9

(i) Draw a diagram showing clearly the forces acting on the parachutis during his free fall.

(ii) Deduce that $\frac{d}{dx}(v^2) = 2g - 2kv^2$

(iii) Show that $v^2 = \frac{g}{k} - Ae^{-2kx}$ satisfies the differential equation in part (ii and show that $A = \frac{g}{k}$

(iv) Sketch the graph of v^2 against x and find an expression for the terminal speed of the parachutist during his free-fall.

End of Exam