



EXTENSION 2 MATHEMATICS

2001 TRIAL EXAMINATION

Time : 3 hours + 5 minutes reading time

Instructions:

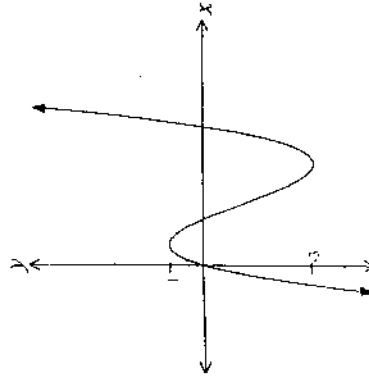
- Attempt all questions
- All questions are of equal value
- All necessary working should be shown for every question.
- Full marks may not be awarded for careless or badly arranged work
- A Table of Standard Integrals is provided
- Approved calculators may be used
- Each question should be answered in a separate booklet

Question 1

- (a) T $(a \cos \theta, b \sin \theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre O. A line through O, parallel to the tangent at T, meets the ellipse at M and N.
- (i) Show the gradient of the tangent at T is $-\frac{b \cos \theta}{a \sin \theta}$ and find the equation of MN. [3]
 - (ii) Show that M and N are $(-a \sin \theta, b \cos \theta)$ and $(a \sin \theta, -b \cos \theta)$ [3]
 - (iii) Show that the area of $\triangle TMN$ is independent of θ . [5]
- (b) Describe the locus $|x - 3| + |x + 3| = 10$ [4]

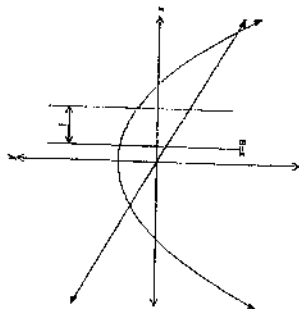
Question 2

(a)



$y = f(x)$ is drawn above. Copy the diagram into your answer booklet and on the same diagram sketch $y = \log_e f(x)$. [2]

- (d) Consider the area between the curves $y = 3 - x^2$ and $y = -2x$. Suppose that two vertical lines 1 unit apart cross this area.



- (i) If the first line is $x = a$, write an expression for the shaded area. [3]
 (ii) Find the maximum value of the shaded area. [2]

Question 4

- (a) Use the substitution $u = x - 1$ to find $\int \frac{x}{\sqrt{x-1}} dx$ [3]
 (b) Find the exact value of (i) $\int_1^5 \log_e x dx$ [2]
 (ii) $\int_0^{\ln 3} e^x \operatorname{cosec}^2(e^x) dx$ [3]
 (c) (i) Using the substitution $u = \frac{1}{x}$, show that $\int_0^1 \frac{\ln x}{1+x^2} dx = \int_0^1 \frac{\ln u}{1+u^2} du$ [2]
 (ii) Deduce the value of $\int_0^1 \frac{\ln x}{1+x^2} dx$ [2]
 (d) Find $\int \frac{\cos x}{\sin x + \sin^2 x} dx$ [3]

- (b) Find the volume of the solid formed when the arc of $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the line $y = 2$ [6]

- (c) A dome has a circular base of radius 10 metres. Cross-sections perpendicular to the base and one axis are parabolas whose height is the same as the base width.

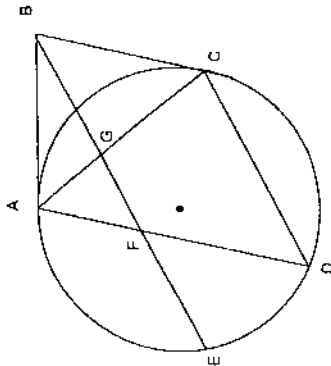
- (i) Why would Simpson's rule give the exact area of the parabolic cross-section? [1]
 (ii) Show that the area of the parabolic cross-section is $\frac{8y^2}{3}$ square metres. [3]
 (iii) Find the volume of the dome. [3]

Question 3

- (a) (i) Express $-1+i$ in modulus argument form [1]
 (ii) Hence evaluate $(-1+i)^{10}$ [2]
 (b) (i) Find all pairs of integers x and y such that $(x+iy)^2 = -3-4i$ [2]
 (ii) Hence or otherwise, solve the quadratic equation $z^2 - 3z + (3+i) = 0$ [2]
 (c) Show, by geometrical means or otherwise that, if z_1 and z_2 are complex numbers such that $|z_1| = |z_2|$, then $\frac{z_1 + z_2}{z_1 - z_2}$ is pure imaginary. [3]

Question 5

(a)



Copy the diagram into your examination booklet

In the diagram EB is parallel to DC. Tangents from B meet the circle at A and C. Prove that

- (i) $\angle BCA = \angle BFA$ [3]
- (ii) ABCF is a cyclic quadrilateral [1]
- (iii) $DF = CF$ [3]
- (b) (i) Draw the graph of $y = \frac{x^4 - 1}{x^2}$ [2]
- (ii) On separate axes sketch $y = \tan^{-1}\left(\frac{x^4 - 1}{x^2}\right)$ [2]
- (c) (i) On the same axes sketch $y = |x| - 2$ and $y = 4 + 3x - x^3$ [2]
- (ii) Hence or otherwise solve $\frac{|x| - 2}{4 + 3x - x^3} > 0$ [2]

Question 6

- (a) Graph the intersection of:
 $z\bar{z} \geq 9$ $z + \bar{z} \leq 8$ $0 < \text{Arg}(z) < \frac{\pi}{4}$ [4]
- (b) Let α be the complex root of the polynomial $z^7 = 1$ with the smallest possible argument.
 Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\delta = \alpha^3 + \alpha^5 + \alpha^6$ [1]
- (i) Explain why $\alpha^7 = 1$ and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$ [1]
- (ii) Show that $\theta + \delta = -1$ and $\theta\delta = 2$ and hence write a quadratic equation whose roots are θ and δ [3]
- (iii) Show that $\theta = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $\delta = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ [2]
- (iv) Write down α in modulus-argument form, and show that $\cos \frac{4\pi}{7} + \cos \frac{2\pi}{7} - \cos \frac{\pi}{7} = -\frac{1}{2}$ and $\sin \frac{4\pi}{7} - \sin \frac{2\pi}{7} - \sin \frac{\pi}{7} = \frac{\sqrt{2}}{2}$ [5]

Question 8

(a) A chord AB and a diameter CD, of a circle centre O, intersect at M within the circle. M is not the centre.

(i) Show that $(CM + MD)^2 > (AM + MB)^2$ [2]

(ii) Deduce that $(CM - MD)^2 > (AM - MB)^2$ [2]

(b) A particle of mass m kg falls from rest in a medium where the resistance to motion is mkv when the particle has velocity v m/s.

(i) Draw a diagram showing the forces acting on the particle. [1]

(ii) Show that the equation of motion of the particle is $\ddot{x} = k(V - v)$ where V m/s is the terminal velocity of the particle in this medium, and x metres is the distance fallen in t seconds. [2]

(iii) Find in terms of V and k the time T seconds taken for the particle to attain 50% of its terminal velocity, and the distance fallen in this time. [4]

(iv) What percentage of its terminal velocity will the particle have attained in time $2T$ seconds? Sketch a graph of v against t showing this information. [3]

(v) If the particle has reached 87.5% of its terminal velocity in 15 seconds, find the value of k . [1]

End of Examination

Question 7

(a) The roots of a cubic equation are α , β and γ , and $\sum \alpha^6 = \alpha^6 + \beta^6 + \gamma^6$ [2]

It is given that $\sum \alpha = -1$, $\sum \alpha^3 = 7$, $\sum \alpha^5 = 8$

(i) Deduce that the equation is $x^3 + x^2 - 3x - 6 = 0$ [2]

(ii) Hence evaluate $\sum \alpha^4$ [2]

(b) (i) If $I_n = \int x(\ln x)^n dx$ for $n \geq 0$, show that $I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}$ [2]

(ii) Hence, find $\int x(\ln x)^2 dx$ [2]

(c) A particle is projected from the origin at an angle of α° with initial velocity V , and it passes through a point (m, n) .

(i) Prove that $gm^2 \tan^2 \alpha - 2mV^2 \tan \alpha + gm^2 + 2nV^2 = 0$ [4]
where g is acceleration due to gravity

(ii) Prove that there are two possible trajectories if

$$(V^2 - gn)^2 > g^2(m^2 + n^2)$$

[3]