ASCHAM SCHOOL

MATHEMATICS EXTENSION 2

TRIAL EXAMINATION

2003

Time : 3 hours + 5 minutes reading time

Instructions:

Attempt all questions

All questions are of equal value

All necessary working should be shown for every question.

Full marks may not be awarded for careless or badly arranged work

A Table of Standard Integrals is provided

Approved calculators may be used

Each question should be answered in a separate booklet

Question 1

a) $(2-3i)(4+i) = p+iq$ where $p,q \in R$. Find p and q. **[1]**

b) (i) Express $z = -\sqrt{3}+i$ in modulus-argument form. **[2]**

(ii) Hence show that $z^7 + 64z = 0$ **[2]**

c) Sketch the following subsets of the Argand diagram, showing important features and intercepts with the axes.

(i) $\{z : 1 < |z| \le 3 \text{ and } 0 < \arg z \le \frac{\pi}{2}\}$ **[2]**

(ii) $\{z : |z+1| + |z-1| = 3\}$ **[3]**

(iii) $\{z : \arg(z-2) - \arg(z+2) = \frac{\pi}{3}\}$ **[2]**

d) Find the Cartesian form of the equation of the locus of the point z if $\operatorname{Re}\left[\frac{z-4}{z}\right] = 0$ **[3]**

Question 2 *Please take a new booklet*

a) Find $\int \frac{e^{2x}}{e^x+1}\,dx$ **[2]**

b) Evaluate $\int \tan^3 x\,dx$ **[2]**

c) Evaluate $\int_0^{\pi} e^x \sin x\,dx$ **[3]**

d) (i) Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{x(\pi-2x)} = \frac{2}{\pi}\ln 2$ **[3]**

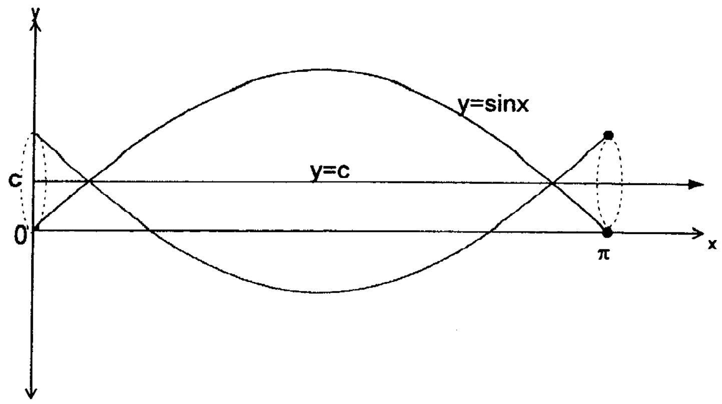
(ii) Using the substitution $u = a+b-x$, show that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ **[2]**

(iii) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi-2x)}\,dx$ **[3]**

Question 3 *Please take a new booklet*

a) A chocolate has a circular base of radius 1 cm. If every section perpendicular to this base is an equilateral triangle, find the volume of chocolate needed to make a box of 40 such chocolates. **[6]**

b)



The arch $y = \sin x$, $0 \le x \le \pi$, is revolved around the line $y = c$ to generate the solid shown.

(i) Show that the volume generated is given by $\pi(\pi c^2 - 4c + \frac{\pi}{2})$ **[6]**

(ii) Find the value of c which minimises the volume. **[3]**

Question 4 *Please take a new booklet*

a) A ball of mass m is thrown vertically upwards under gravity, the air resistance to the motion being $\frac{mgv^2}{a^2}$ where the speed is v, a is a constant and g is the acceleration due to gravity.

(i) Show that during the upward motion of the ball

$$v\frac{dv}{dx} = \frac{-g}{a^2}\left(a^2 + v^2\right)$$

where x is the upward displacement. **[2]**

(ii) Show that the greatest height reached is $\frac{a^2}{2g}\ln\left(1+\frac{u^2}{a^2}\right)$ where u is the speed of projection. **[5]**

b) A curve is defined by the parametric equations $x = \cos^3\theta$, $y = \sin^3\theta$ for $0 < \theta < \frac{\pi}{4}$.

(i) Show that the equation to the normal to the curve at the point P $(\cos^3\phi, \sin^3\phi)$ is $x\cos\phi - y\sin\phi = \cos 2\phi$ **[4]**

(ii) The normal at P cuts the x-axis at A and the y-axis at B. Show $AB = 2cot2\phi$ **[4]**

Question 5 *Please take a new booklet*

a) If $ax^3 + bx^2 + d = 0$ has a double root, show that $27a^2d + 4b^3 = 0$ **[3]**

b) (i) Prove that $P(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x + c$ has no real zeros if $c > 9\frac{1}{3}$ **[4]**

(ii) Explain why the largest zero of $P(x)$ is greater than 2 if $c = -2$. Find an approximation for the largest zero of $P(x)$ using one application of Newton's method. **[3]**

c) (i) P is any point inside a circle center O. M is the midpoint of chords AB through P. Find the locus of M. Explain your answer. **[3]**

(ii) Q is any point outside a circle center C. N is the midpoint of chords DE through Q. State the locus of N. **[2]**

Continued on next page

Question 6 *Please take a new booklet*

a) (i) Find the five fifth roots of unity. **[2]**

(ii) If $\omega = cis\frac{2\pi}{5}$, show that $1+\omega+\omega^2+\omega^3+\omega^4=0$ **[3]**

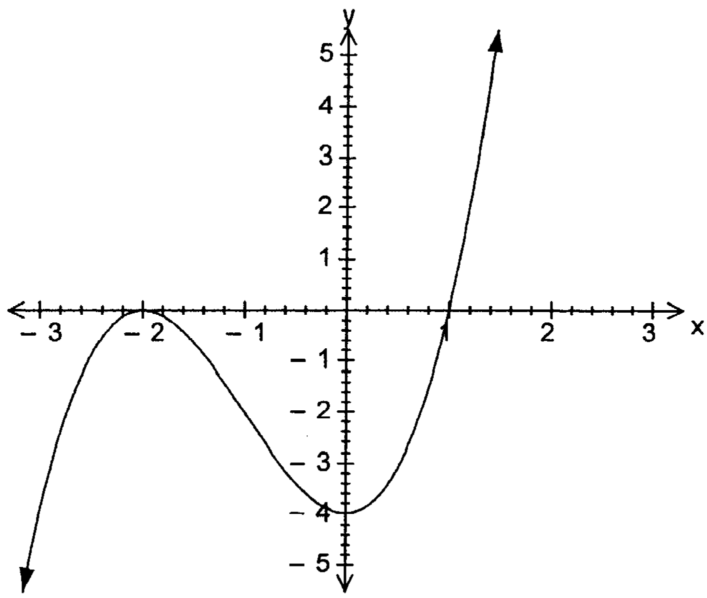
(iii) Show that $z_1=\omega+\omega^4$ and $z_2=\omega^2+\omega^3$ are roots of the equation $z^2+z-1=0$ **[3]**

b) (i) By using the expansions of $\cos(x-y)$ and $\cos(x+y)$ show that $\sin x\sin y=\frac{1}{2}(\cos P-\cos Q)$ where $P=(x-y)$ and $Q=(x+y)$ **[3]**

(ii) Hence prove that $\sin x+\sin 3x+\sin 5x+........+\sin(2n-1)x=\dfrac{\sin^2 nx}{\sin x}$ **[4]**

Question 7 *Please take a new booklet*

a) The graph of $y=x^3+3x^2-4$ is sketched below



(i) Sketch the curves $y=\left|x^3+3x^2-4\right|$ and $y=\ln\left|x^3+3x^2-4\right|$ on separate axes. **[3]**

(ii) Hence or otherwise determine the value of m, where m is a constant, such that the equation $2\ln|x+2|+\ln|x-1|=m$ **[4]**

b) AB is a diameter of a circle whose centre is O and C is a point on the circumference such that $\angle AOC$ is acute. OC is produced to meet the tangent at A in D. Let $\angle CBD = \alpha$ and $\angle ABC = \beta$. Prove

(i) $\tan(\alpha+\beta)=\frac{1}{2}\tan 2\beta$ **[3]**

(ii) $\tan\alpha=\tan^3\beta$ **[3]**

(iii) Calculate the value of α when AD = AB **[2]**

Question 8 *Please take a new booklet*

a) (i) Show that the condition for the line $y=mx+c$ to be tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$ **[3]**

(iii) Hence or otherwise prove that the pair of tangents from the point (3, 4) to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ are at right angles to each other. **[4]**

b) Let $I_{2n}=\int_{-1}^{1}(1-x^2)^n dx$ where $n\geq 0$

(i) Use the substitution $x=\sin\theta$ to show that $I_{2n}=\frac{2n}{2n+1}I_{2n-2}$ **[3]**

(ii) Show that $I_6=\frac{32}{35}$ **[2]**

(iii) Deduce that $I_{2n}=\frac{2^{2n+1}(n!)^2}{(2n+1)!}$ **[3]**

End of Examination