Ex1, 2

Question 1

(a) Find (i) $\int \frac{dx}{1+\sin x}$ using $\tan\frac{x}{2}=t$ **3m**

(ii) $\int \frac{dx}{(x+1)(x+3)^2}$ **3m**

(b) By rationalising the numerator or otherwise, evaluate

$$\int_0^5 \sqrt{\frac{5-x}{5+x}}\,dx$$ **4m**

(c) Let $I_n=\int_0^{\frac{\pi}{2}} \sin^n x\,dx$, where n is a non negative integer

(i) Show that $I_n=(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\cos^2 x\,dx$ for $n\geq 2$ **2m**

(ii) Deduce $I_n=\frac{n-1}{n}I_{n-2}$, where $n\geq 2$ **2m**

(iii) Evaluate I_4 **1m**

Question 2

(a) Let $z=-5-12i$, find in the form $a+ib$

(i) $\frac{1}{z}$ **2m**

(ii) The square roots of z **3m**

(b) Let $\omega=b-ib\sqrt{3}$, where b is a positive real number

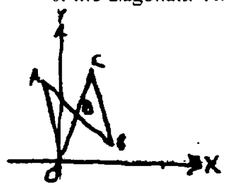
(i) Find $|\omega|$ and $\arg\omega$ **2m**

(ii) Hence find ω^5 in the form $x+iy$ **2m**

(c) On the Argand diagram, sketch the locus defined by

$$(z-1)(\bar{z}-1)=1$$ **3m**

(d) OACB is a rectangle, where OA is twice OB. D is the point of intersection of the diagonals. The point B represents the complex number z.



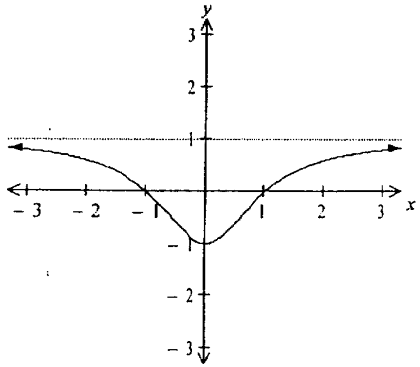
Find in terms of z, the complex number represented by

(i) A **1m**

(ii) D **2m**

Question 3

(a)



The diagram shown is the graph of $y=f(x)$, where $f(x)=\frac{x^2-1}{x^2+1}$

(i) On the separate diagrams provided on pages 10 and 11 sketch the following graphs, showing any intercepts on the coordinate axes and the equations of any asymptotes. Detach the page and insert in the answer booklet to Question 3 **8m**

α) $y=[f(x)]^2$

β) $y=\sqrt{f(x)}$

γ) $y=\frac{1}{f(x)}$

δ) $y=e^{f(x)}$

(ii) The function $f(x)$ with its domain restricted to $x\geq 0$ has an inverse $f^{-1}(x)$. Find $f^{-1}(x)$ as a function of x. Sketch $f^{-1}(x)$ on the axis provided on page 12 and insert in your answer booklet. **4m**

(b) If $P(x)=x^4-3x^3+2x^2+ax+b$ is divisible by $(x-1)^2$ find the values of a and b **3m**

Question 4

(a) $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

M is the midpoint of SP, where S is a focus of the ellipse.

(i) Find the coordinates of M **1m**

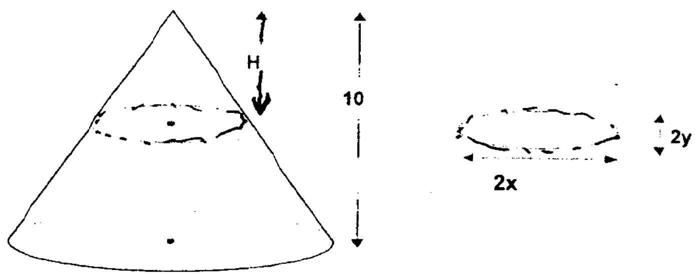
(ii) Find the cartesian equation of the locus of M **3m**

(iii) Prove that the locus is a second ellipse with centre at the midpoint of OS, where O is the origin. **2m**

(b) (i) Evaluate $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$

(ii) Explain how you could use part (b)(i) to prove that the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab $units^2$ **2m**

(iii)



The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ forms the base of a cone of height 10 units.

A slice ΔH wide is taken H units from the vertex as shown. The cross-section is an ellipse with major and minor axes $2x$ and $2y$ respectively.

Use the result from b(ii) to prove that the area of the cross-section H units from the vertex is $\frac{\pi abH^2}{100}$ $units^2$ **4m**

(iv) Hence find the volume of the elliptical cone. **2m**

Question 5

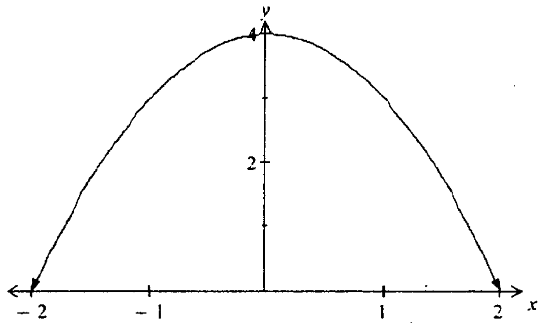
(a) If $z=\cos\theta+i\sin\theta$

(i) Show that $z^n+z^{-n}=2\cos n\theta$

(ii) Hence or otherwise, show that

$$\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$$ **4m**

(b) The solid shown has a semicircular base of radius 2 units. The vertical cross sections perpendiculat to the diameter are right angled triangles whose height is bound by the parabola $z=4-x^2$



By slicing at right angles to the x-axis, show that the volume of the solid is given by $V=\int_0^2(4-x^2)^{\frac{3}{2}}\,dx$ and hence calculate the volume. **5m**

(c) Prove $\sin 3\theta=3\sin\theta-4\sin^3\theta$ **2m**

(d) In ΔABC, $\angle A=2\angle B$, $\angle C$ is obtuse and the three lengths a, b, c are integers,

(i) Show $2\cos B=\frac{a}{b}$

(ii) Show that $a^2=b(b+c)$ **4m**

Question 6

(a) A particle is projected from the origin with speed V at an angle α to the horizontal. The particle is subject to both gravity and an air resistance proportional to its velocity, so that its respective horizontal and vertical components of acceleration while it is <u>rising</u> are given by

$\ddot{x} = -k\dot{x}$

$\ddot{y} = -g - k\dot{y}$

(i) Show that **4m**

α) $\dot{x} = (V\cos\alpha)e^{-kt}$

β) $\dot{y} = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}$

(ii) Hence show that $x = \frac{V\cos\alpha}{k}\left(1 - e^{-kt}\right)$ **2m**

(iii) When the particle has reached its greatest height, show that it has travelled a horizontal distance of

$$\frac{V^2\sin 2\alpha}{2(g + Vk\sin\alpha)}$$ **2m**

(b) The polynomial $P(x) = x^3 + cx + d$, where c and d are real and non zero, has zeroes $a+ib,\ a-ib$ and k, where a, b are real and non zero and $k < 0$.

It is known that the graph of $y = P(x)$ has two turning points.

(i) By considering $P'(x)$, show that $c < 0$ **1m**
(ii) Sketch the graph of $y = P(x)$ **2m**
(iii) Deduce that $a > 0$ **1m**
(iv) Show that $d = 8a^3 + 2ac$ **3m**

Question 7

(a) For what values of k does the equation

$$\frac{x^2}{29-k} + \frac{y^2}{4-k} = 1$$

represent (i) an ellipse
(ii) an hyperbola **2m**

(b) Show that the foci of each ellipse in a(i) is independent of k **2m**

(c) A sequence u_1, u_2, u_3.......... is defined by

$$u_1 = 1,\ u_2 = 7\ \ and\ \ u_n = 7u_{n-1} - 12u_{n-2}\ \ \text{for}\ \ n \geq 3$$

Use the method of mathematical induction to show that

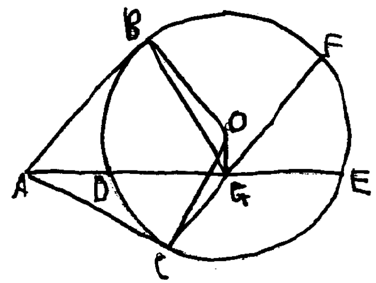
$$u_n = 4^n - 3^n\quad for\ \ n \geq 1$$ **7m**

d) Using the fact that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **4m**

(i) Solve $8x^3 - 6x - 1 = 0$
(ii) Deduce that $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$

Continued......

Question 8
(a)



In the diagram, AB and AC are the tangents from A to the circle with centre O, meeting the circle at B and C. ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the circle at F.

Using the sheets on the pages 8 and 9 with the diagram redrawn for you, complete the questions below and insert the sheets ,with your name on them, into your answer booklet.

(i) Show that ABOC and AOGC are cyclic quadrilaterals **4m**
(ii) Hence prove that BF is parallel to AE **4m**
(hint: Construct BC, BF and let $\angle ABC = \theta$)

(b)
(i) Prove for all $a, b > 0$, $a^2 + b^2 \geq 2ab$ **1m**

(ii) Hence prove that for all $a, b, c > 0$ **2m**
$a^2 + b^2 + c^2 \geq bc + ca + ab$

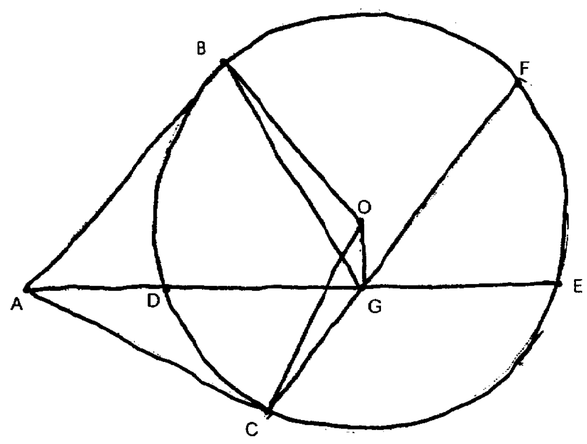
(iii) Given the identity
$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$
Prove that for all $a, b, c > 0$, $a^3 + b^3 + c^3 \geq 3abc$ **2m**

(iv)Using a suitable substitution into (iii), show that for all $x, y, z > 0$
$x + y + z \geq 3(xyz)^{\frac{1}{3}}$ **2m**

Name................................Teacher's Name...........

Question 8a

Show working on the diagram



More space for Question 8 pto