

Barker College Unit maths trial 1999

Question 1 (Start a new page)

MARKS

a) Find

3

i) $\int \frac{4}{x^2 + 4} dx$

ii) $\int \frac{4}{\sqrt{x^2 + 4}} dx$

iii) $\int \frac{4x}{\sqrt{x^2 + 4}} dx$

b) Evaluate

8

i) $\int_0^{\frac{\pi}{2}} e^x \cos x dx$

ii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + \cos \theta}$

c)

4

i) Find polynomials $p(x), q(x)$ of degrees less than 2, such that $(x+2)p(x) + (x^2+4)q(x) = 1$.

ii) Hence evaluate $\int_0^2 \frac{8dx}{(x+2)(x^2+4)}$.

Question 2 (Start a new page)

MARKS

a) If $z = \frac{3+2i}{1-2i}$ then find

3

i) \bar{z}

ii) $\arg z$

b)

5

i) Express $\sqrt{6i-8}$ in the form $a+ib$ where a, b are elements of the set of reals.

ii) Hence solve $2z^2 - (3+i)z + 2 = 0$ for z . Express your answer in the form $a+ib$.

c) Neatly sketch each of the following loci on separate Argand Diagrams.

4

i) $\arg \frac{z+1}{z-i} = \frac{2\pi}{3}$

ii) $z\bar{z} = z + \bar{z}$

d)

3

i) Show on an Argand diagram the locus of z where $|z-4-3i|=1$.

ii) What are the least values of $|z|$.

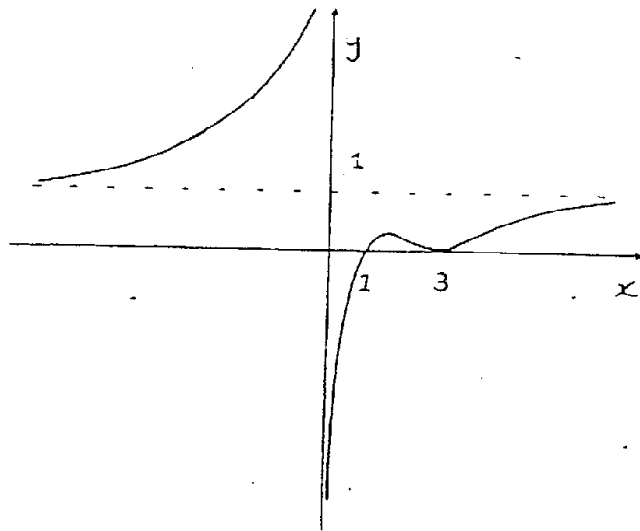
Question 3 (Start a new page)

- a)
- i) Sketch $y = f(x)$, clearly labelling all essential features given that $f(x) = x^3 - 4x$.

On separate diagrams sketch showing labelling all essential features

- ii) $y^2 = f(x)$
- iii) $y = f\left(\frac{1}{x}\right)$
- iv) $y = e^{f(x)}$
- v) $|y| = |f(x)|$

b)



The diagram above is of the derivative of $y = f(x)$. i.e. The curve has equation $y = f'(x)$.

- i) Sketch the function $y = f''(x)$.
- ii) On a separate diagram sketch a possible graph of $y = f(x)$.
- iii) Suggest a possible equation for $y = f'(x)$ in terms of x .

Question 4 (Start a new page)

MARKS

a) 8

Show that the normal to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ bisects the angle between the lines $x = 2ap$ and SP where S is the focus of the parabola.

b) 7

i) Sketch the hyperbola with equation $\frac{x^2}{4} - \frac{y^2}{2} = 1$, carefully labelling all essential features.

ii) Show that the equation of the tangent to this hyperbola at $P(2 \sec \theta, \sqrt{2} \tan \theta)$ is given by $\frac{x \sec \theta}{2} - \frac{y \tan \theta}{\sqrt{2}} = 1$.

iii) Hence prove that the area of the triangle bounded by this tangent and the asymptotes of the hyperbola is independent of the position of P .

Question 5 (Start a new page)

N

a)

i) Prove that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has a magnitude of πab .

ii) Find the volume of a mound with a circular base of equation $x^2 + y^2 = 4$ which has semi-elliptical cross-sections parallel to the y axis, where the ratio of the major axis : minor axis = 2 : 1. The height of each cross-section is the length of the semi-minor axis.

b)

i) Sketch the curve $y = x^2(x^2 - 1)$ shading the region bounded by the curve and the x -axis.

ii) Find the volume of the solid formed when this shaded area in part i) is rotated about the y -axis.

iii) What is the volume of the solid formed when the area encompassed by the relation $y^2 = x^8 - 2x^6 + x^4$ is rotated about the y -axis?

Question 6 (Start a new page)

MARKS

- a) Show that $1+i$ is a root of the polynomial $P(x) = x^3 + x^2 - 4x + 6$ and hence completely factorize $P(x)$ over the field of complex numbers. 3
- b) 4
- i) If the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has roots of the form $a+ib$ and $a-2ib$ where a, b are real, find the values of a and b .
- ii) Find all the zeros of $P(x)$.
- iii) Express $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ as a product of two quadratic factors with rational coefficients.
- c) 5
- i) Prove that if the polynomial $P(x)$ has a root α of multiplicity m then $P'(x)$ has a root α of multiplicity $m-1$.
- ii) Given that the polynomial $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a root of multiplicity 3, find all the roots of $P(x)$.
- d) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, prove that $(\beta - \gamma)^2 + (\alpha - \beta)^2 + (\alpha - \gamma)^2 = -6q$. 3

Question 7 (Start a new page)

a) If $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}} dx$ where $n = 0, 1, 2, 3, \dots$

i) Show that $x^{n-1} \sqrt{x+1} = \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}}$.

ii) Show that $(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$ for $n = 1, 2, 3, \dots$

iii) Evaluate $\int_0^1 \frac{x^2}{\sqrt{x+1}} dx$.

b)

i) Sketch on an argand diagram the roots of $z^5 - 1 = 0$.

ii) Show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$.

iii) Hence or otherwise find the exact values of $\cos \frac{2\pi}{5}$ and $\cos \frac{\pi}{5}$.

Question 8 (Start a new page)

MARKS

a) Prove that if the opposite angles of a quadrilateral are supplementary then the quadrilateral must be cyclic. 4

b) 5

i) Show that $\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab}$.

ii) Simplify $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c$.

iii) If the equation $x^3 - 2x^2 - 5x + 4 = 0$ has roots α, β, γ show that $\tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma = \frac{\pi}{4}$.

c) 6

i) Show that $f(x) = \frac{\sec x + \tan x}{2 \sec x + 3 \tan x}$ is a decreasing function in term of x for the domain $0 < x < \frac{\pi}{2}$.

ii) Deduce that $\frac{\pi}{28} > \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec x + \tan x}{2 \sec x + 3 \tan x} dx > (\sqrt{2} - 1) \frac{\pi}{12}$.

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