YEAR 12 – TRIAL 2006 – EXTENSION 1

QUESTION 1

MARKS

a) Find $\dfrac{d}{dx}(e^{2x}\cos 3x)$ — 2

b) Evaluate — 2

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-3x^2}}\,dx$$

c) Evaluate $\lim\limits_{x \to 0} \dfrac{x}{\sin 4x}$ — 1

d) Use the substitution $u = 1 + x^4$ to evaluate — 2

$$\int_0^1 \frac{x^3}{1+x^4}\,dx$$

e) If the roots of the equation $x^3 + 2x^2 - x - p = 0$ are c, c + 1 and c + 3, find the value of p. — 2

f) A (1, 0) and B (2, 4) are two points in the number plane. — 1

Find the coordinates of the point P that divides the interval AB externally in the ratio 3:1.

g) Find the term independent of x in the expansion $\left(x + \dfrac{2}{x}\right)^{10}$ — 2

QUESTION 2 **MARKS**

a) Solve $\dfrac{3}{2-x} \geq 1$ 2

b) Consider the polynomial 2

$P(x) = (x+2)(x-3)\,Q(x) + ax + b$

Given that $P(x)$ has remainders 1 and 6 when divided by $(x+2)$ and $(x-3)$ respectively, find a and b.

c) Consider the function

$$y = \pi + 2\sin^{-1}\left(\frac{x}{3}\right)$$

i) Find the domain and the range. 2

ii) Sketch the graph of the function. 2

d) The region bounded by the curve $y = \sin^{-1} x$, the y axis and $y = \dfrac{\pi}{6}$, is rotated about the y axis to form a solid.

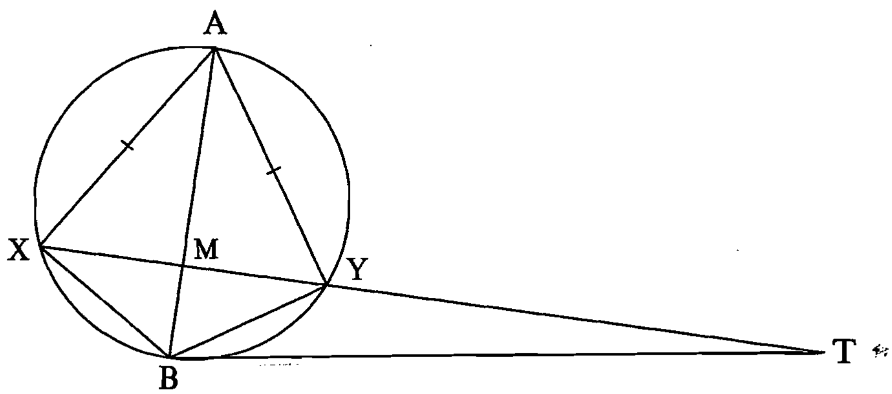
i) Show that the volume of the solid obtained is given by 2

$$V = \pi \int_0^{\frac{\pi}{6}} \sin^2 y \, dy$$

ii) Find the volume of the solid. 2

QUESTION 3 **MARKS**

a) A, X, B and Y are points on the circumference of a circle such that AX = AY.
XY meets AB at the point M.
The tangent at B meets the chord XY produced at T.



i) Explain why $\angle ABY = \angle AYX$. 1

ii) Show that AB bisects $\angle XBY$. 1

iii) Show that BT = MT. 2

b) Consider the functions $f(x) = 4 - x^2$ and $g(x) = \ln x$.

i) Sketch the graphs of f(x) and g(x) on the same set of axes for $x > 0$. 2

ii) Use your graph to show that the equation $\ln x + x^2 - 4 = 0$ has only one root which is near $x = 1.5$ 1

iii) Use one application of Newton's method to find a better approximation of the root of the equation $\ln x + x^2 - 4 = 0$ 2

c) Ten people are to be seated around two circular tables. Six of them can sit at one table and the remaining four can sit at the other table.

i) How many different groups can be formed to sit around these two tables? 1

ii) How many seating arrangements are possible around the two tables? 2

QUESTION 4 **MARKS**

a) The population N of a particular species of birds in an island at any time t is expressed as

$N = 2000 + c\,e^{-kt}$,where c and k are constants.

Given that the initial population was 11 000 birds and it decreased to 8 000 after 10 years,

i) find the constants c and k. 2

ii) find the time required for the population to decrease to 6 000 birds. 2

b) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The tangents to the parabola at P and Q intersect at an angle of 60° at a point A.

i) Show that the coordinates of A are $\left(a(p+q)\,,\, apq\right)$ 2

(You may assume that the equation of the tangent at P is $y = px - ap^2$)

ii) Show that $p - q = \sqrt{3}\,(1 + pq)$ 1

iii) Show that as P and Q moves on the parabola the point A moves on the curve with equation $x^2 = 3a^2 + 10ay + 3y^2$ 2

c) 75% of the workers in a shoe factory are highly skilled, while 25% are less skilled.

Each worker makes the same number of pairs of shoes a day. For highly skilled workers only 1% of the pairs of shoes are defective, and 2% are defective for the less skilled.

i) What is the probability that a pair of shoes made in this factory is defective? 1

ii) What is the probability that, in a random parcel of 20 pairs of shoes, no more than one pair is defective? 2

QUESTION 5 **MARKS**

a) Let $g(x) = \dfrac{x}{1+x^2}$ for all real values of x.

i) Sketch the graph of g(x) showing the coordinates of the turning points and the x and y intercepts. 3

ii) What is the largest domain containing the value $x = 2$ for which g(x) has an inverse function $g^{-1}(x)$? 1

iii) Sketch the graph of the inverse function $y = g^{-1}(x)$ on the same set of axes as your graph in part (i). 1

iv) Find an expression for $y = g^{-1}(x)$ in terms of x. 1

b) A particle moves in a straight line and its displacement x centimetres from the origin at time t seconds is given by :

$$x = 2 + 4\cos^2\left(\frac{\pi}{2}t\right)$$

i) Show that the motion of the particle is simple harmonic. 2

ii) Find the maximum velocity of the particle. 1

c) Use mathematical induction to prove that, for all integers n with $n \geq 1$. 3

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

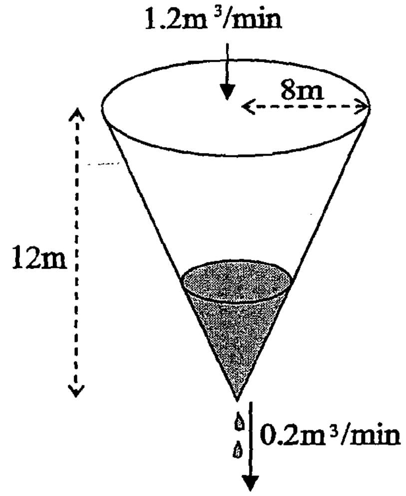
QUESTION 6 **MARKS**

a) Consider the function $f(x) = \cos^{-1}\left(\frac{1-x}{1+x}\right) - 2\tan^{-1}\sqrt{x}$ where $x > 0$.

i) Show that $f'(x) = 0$ 3

ii) Sketch the graph of $y = f(x)$ 1

b) A tank has the shape of an inverted right circular cone of base radius 8m and height 12m contains water to a depth 3 m



Water starts to be poured into the tank at the constant rate of 1.2m^3/min but at the same time a leak at the vertex of the tank causes a discharge of the water at a constant rate of $0.2\ \text{m}^3$/min.

i) Show that the volume V of the water in cubic metres at any time t can be expressed as 2

$V = 4\pi + t$

ii) Calculate the rate at which the height is increasing at the time $t = 28\pi$ minutes. 3

iii) Calculate the rate at which the area of the water in contact with the wall of this tank is increasing at the time $t = 104\pi$ minutes. 3

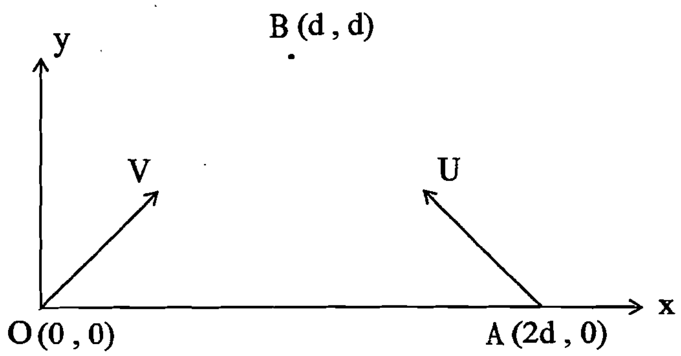
QUESTION 7

MARKS

a) Given that $(1+\frac{1}{x})^n(1+x)^n=\frac{1}{x^n}(1+x)^{2n}$, show that 3

$${}^nC_0\,{}^nC_3+{}^nC_1\,{}^nC_4+\ldots+{}^nC_{n-3}\,{}^nC_n=\frac{(2n)!}{(n+3)!(n-3)!}$$

b)



Two projectiles are projected at the same time towards a target at point B(d,d). One of the projectiles is projected from O(0,0) with a velocity V, while the second projectile is projected from A(2d,0) with velocity U.

At the instant when the projectiles are projected target B starts falling vertically down under gravity.

Consider the axes as shown and assume that there is no air resistance, and that g is the acceleration due to gravity.

i) Show that the equations of the positions of the target are 1

$y=-\frac{1}{2}gt^2+d$ and $x=d$

ii) Find the equations of the positions of the two projectiles. 3

iii) The projectile from O hits the target in the air. 3
Find the time taken by the projectile to hit the target and show that $V>\sqrt{gd}$

iv) After the impact a fragment of the target falls vertically down under gravity and t minutes later the particle from A hits this fragment. 2

Show that $t=\sqrt{2}\,d\left(\frac{1}{U}-\frac{1}{V}\right)$

MR. PERCH

YEAR 12 – TRIAL 2006 – EXTENSION 1 – SOLUTIONS

Question 1:

a) $\frac{d}{dx}\left(e^{2x}\cos 3x\right)$

Using the product rule:

$u = e^{2x} \quad v = \cos 3x$

$u' = 2e^{2x} \quad v' = -3\sin 3x$

$\therefore \quad = 2e^{2x}.\cos 3x - 3e^{2x}\sin 3x$

$= e^{2x}(2\cos 3x - 3\sin 3x)$ (2 marks)

b) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-3x^2}}\,dx$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{3}\sqrt{\frac{1}{3}-x^2}}\,dx$

$= \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{x}{\frac{1}{\sqrt{3}}}\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\left[\sin^{-1}\sqrt{3}x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right]$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - -\frac{\pi}{3}\right) = \frac{2\pi}{3\sqrt{3}}$ (2 marks)

c) $\lim_{x\to 0} \frac{x}{\sin 4x} = \lim_{x\to 0} \frac{1}{4} \times \frac{4x}{\sin 4x}$

$= \lim_{x\to 0} \frac{1}{4} \times \frac{1}{\frac{\sin 4x}{4x}} = \frac{1}{4} \times \frac{1}{1} = \frac{1}{4}$ (1 mark)

d) $I = \int_0^1 \frac{x^3}{1+x^4}\,dx$

let $u = 1 + x^4$

$\therefore \frac{du}{dx} = 4x^3$

$\therefore \frac{1}{4}du = x^3\,dx$

when $x = 1$, $u = 2$

when $x = 0$, $u = 1$

$\therefore I = \frac{1}{4}\int_1^2 \frac{1}{u}\,du = \frac{1}{4}\left[\ln u\right]_1^2$

$= \frac{1}{4}(\ln 2 - \ln 1) = \frac{1}{4}\ln 2$ (2 marks)

e) $x^3 + 2x^2 - x - p = 0$

The roots are c, $c+1$, $c+3$.

The sum of the roots is $c+c+1+c+3 = -2$

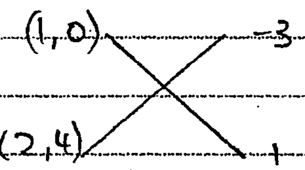
$\therefore 3c + 4 = -2$, $3c = -6$, $c = -2$

The products of the roots is

$c(c+1)(c+3) = p$, but $c = -2$

$\therefore -2(-1)(1) = p \quad \therefore p = 2$ (2 marks)

f) $A(1,0)$ $B(2,4)$



$x = \frac{(1\times 1) + (-3\times 2)}{-3+1}$

$= \frac{1-6}{-2} = 2\frac{1}{2}$

$y = \frac{(1\times 0) + (-3\times 4)}{-3+1} = \frac{-12}{-2} = 6$

$\therefore$ The point P is $\left(2\frac{1}{2}, 6\right)$ (1 mark)

g) $\left(x + \frac{2}{x}\right)^{10}$

$T_{r+1} = {}^{10}C_r\, x^{10-r}\left(\frac{2}{x}\right)^r$

$= {}^{10}C_r\, x^{10-r}\, 2^r\, x^{-r}$

$= {}^{10}C_r\, 2^r\, x^{10-2r}$

To find the term independant of x, we let the power of x equal zero.

$\therefore 10 - 2r = 0$, $r = 5$

$\therefore T_6 = {}^{10}C_5\, 2^5 = 8064$ (2 marks)

Question 2:

a) $\frac{3}{2-x} \geq 1$

$\frac{3}{2-x} - 1 \geq 0$, $\frac{3-(2-x)}{2-x} \geq 0$

$\frac{3-2+x}{2-x} \geq 0$, $\frac{1+x}{2-x} \geq 0$

x		-1		2	
x	$-$	0	$+$		$+$
$-x$	$+$		$+$	•	$-$
ult	$-$	0	$+$	•	$-$

$\therefore$ solutions are $-1 \le x < 2$. (2 marks)

) $P(x) = (x+2)(x-3)\,Q(x) + ax + b$

s $P(-2) = 1$ and $P(3) = 6$, by

ubstitution we get $-2a + b = 1$

$\therefore 2a - b = -1$ ①

$3a + b = 6$ ②

By solving these equations

imultaneously we get $5a = 5$

$\therefore a = 1$. $3 + b = 6$, $\therefore b = 3$

(2 marks)

) $y = \pi + 2\sin^{-1}\left(\frac{x}{3}\right)$

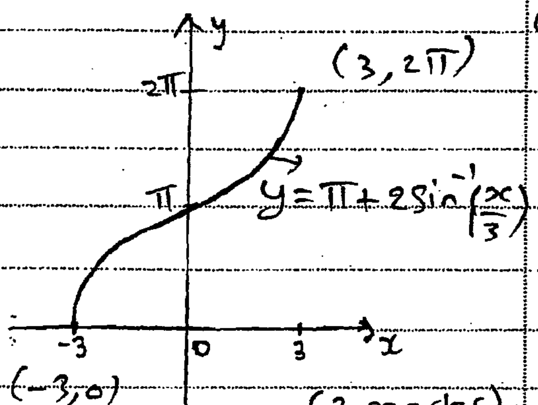
Domain: $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$.

ange: $\frac{-\pi}{2} \le \sin^{-1}\left(\frac{x}{3}\right) \le \frac{\pi}{2}$

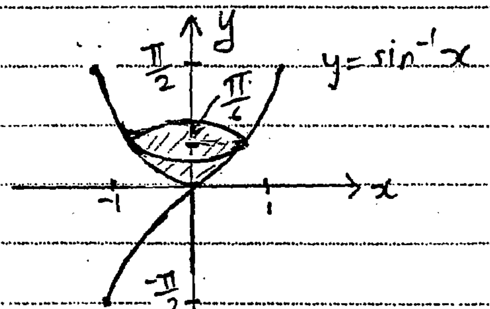
$-\pi \le 2\sin^{-1}\left(\frac{x}{3}\right) \le \pi$

$\therefore 0 \le y \le 2\pi$. (2 marks)



(2 marks)

d)



i) $y = \sin^{-1}x$ $\therefore x = \sin y$ $\therefore x^2 = \sin^2 y$

Volume formed when the area is rotated about the y axis is

$V = \pi \int_a^b x^2\,dy$ $\therefore V = \pi \int_0^{\frac{\pi}{6}} \sin^2 y\,dy$. (2 marks)

ii) $\cos 2y = 1 - 2\sin^2 y$

$2\sin^2 y = 1 - \cos 2y$

$\sin^2 y = \frac{1}{2}(1 - \cos 2y)$

$\therefore V = \frac{\pi}{2} \int_0^{\frac{\pi}{6}} (1 - \cos 2y)\,dy = \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4} - (0 - 0)\right] = \frac{\pi^2}{12} - \frac{\sqrt{3}\pi}{8}$ units^3 (2 marks)

Question 3:

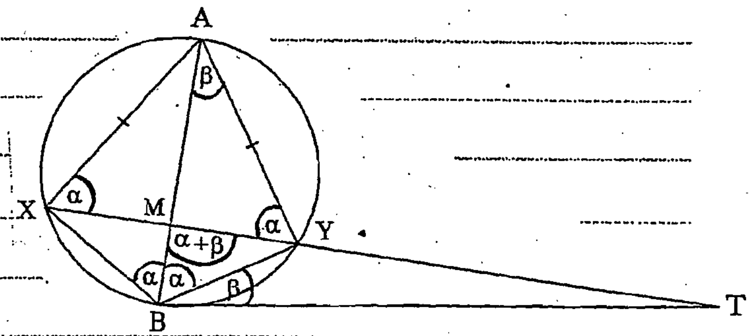
a) Data: $AX = AY$

Aim: (i) Explain why $\angle ABY = \angle AYX$

(ii) Show that $AB \angle XBY$.

(iii) Show that $BT = MT$.

Construction:

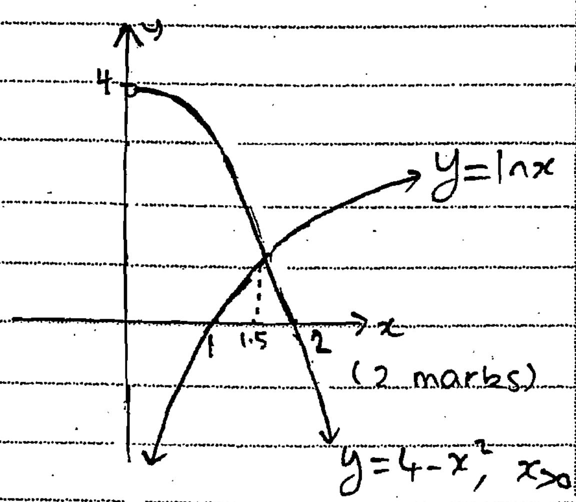


Proof:

i) Let $\angle ABY = \alpha$ $\therefore \angle AXY = \alpha$ (angles at the circumference subtended by the same arc AY are equal).

$AY = AY$ (given)
$\therefore \Delta AXY$ is isosceles (2 equal sides)
$\therefore \angle AYX = \angle AXY = \alpha$ (base angles in isosceles triangle AXY equal)
$\therefore \angle ABY = \angle AYX = \alpha$ (1 mark)
ii) $\angle AYX = \angle ABX = \alpha$ (angles at the circumference subtended by same arc AX equal)
$\therefore \angle ABY = \angle ABX = \alpha$
Hence, AB bisects $\angle XBY$ (1 mark)
iii) Let $\angle TBY = \beta$
$\therefore \angle BAY = \beta$ (angle in the alternate segment, tangent TB and chord BY)
$\therefore \angle BMT = \alpha + \beta$ (exterior angle of ΔAMY equals the sum of the opposite interior angles)
$\angle MBT = \angle MBY + \angle YBT = \alpha + \beta$ (adj $\angle$'s)
$\angle BMT = \angle MBT = \alpha + \beta$
ΔBMT is isosceles (base angles =)
$BT = MT$ (2 equal sides in isos ΔBMT) (2 marks)

i) $f(x) = 4 - x^2 \quad g(x) = \ln x$



(2 marks)

ii) From the graph, $y = \ln x$ and $y = 4 - x^2$ intersect only once near $x = 1.5$.
$\therefore$ The equation $\ln x = 4 - x^2$ (i.e. $\ln x + x^2 - 4 = 0$) has a root close to $x = 1.5$. (1 mark)
iii) Let $h(x) = \ln x + x^2 - 4$
$\therefore h'(x) = \frac{1}{x} + 2x$
$\therefore h(1.5) = \ln(1.5) + (1.5)^2 - 4 = -1.3445\ldots$
$\therefore h'(1.5) = \frac{1}{1.5} + (2 \times 1.5) = 3.666\ldots = 3\frac{2}{3}$
$\therefore x_1 = 1.5 - \frac{h(1.5)}{h'(1.5)} = 1.866\ldots \doteqdot 1.87$ (2dp)
$\therefore h(1.87) = 0.1228\ldots$
$\therefore x = 1.87$ is a better approximation to the root. (2 marks)

c) i) There are ${}^{10}C_6$ ways of selecting 6 people for the first table, leaving ${}^4C_4$ ways of selecting 4 people from the remaining 4, for the second table.
$\therefore {}^{10}C_6 \times {}^4C_4 = 210$ different groups can be selected. (1 mark)
ii) For each group in part (i) there are 5! arrangements for one table and 3! arrangements for the other.
$\therefore$ Total is $5! \times 3! \times 210 = 151200$ (2 marks)

Question 4:

a) i) $N = 2000 + ce^{-kt}$
When $t = 0$, $N = 11000$
$\therefore 11000 = 2000 + ce^0$
$\therefore c = 9000$
$\therefore N = 2000 + 9000e^{-kt}$
When $t = 10$, $N = 8000$
$\therefore 8000 = 2000 + 9000e^{-10k}$

$00 = 9000e^{-10k}$

$- = e^{-10k}$

plying $\log_e$ to both sides

$(\frac{2}{3}) = -10k \ln e$

$k = \frac{-1}{10} \ln(\frac{2}{3})$

$N = 2000 + 9000e^{\frac{1}{10}\ln(\frac{2}{3})t}$ (2 marks)

$N=6000$

$6000 = 2000 + 9000e^{\frac{1}{10}\ln(\frac{2}{3})t}$

$000 = 9000e^{\frac{1}{10}\ln(\frac{2}{3})t}$

$- = e^{\frac{1}{10}\ln(\frac{2}{3})t}$

plying $\log_e$ on both sides

$(\frac{4}{9}) = \frac{1}{10}\ln(\frac{2}{3})t \ln e$

$t = \frac{\ln(\frac{4}{9})}{\frac{1}{10}\ln(\frac{2}{3})} = 20$ years (2 marks)

) The equation of the tangent at s $y = px - ap^2$ (given).

ierefore, the equation of the tangent at Q is $y = qx - aq^2$.

subtraction $(p-q)x - a(p^2-q^2) = 0$

$-q)x = a(p-q)(p+q)$

$= a(p+q) \quad (p \neq q)$

$y = p[a(p+q)] - ap^2$

$= ap^2 + apq - ap^2$

$= apq$

$A\,[a(p+q),\ apq]$ (2 marks)

The gradient at P is p. The dient at Q is q. The lines ersect at an angle of 60°

$\text{an } 60° = \frac{m_1 - m_2}{1 + m_1 m_2}$

$\text{n } 60 = \frac{p-q}{1+pq} \quad \therefore \sqrt{3} = \frac{p-q}{1+pq}$

so $p-q = \sqrt{3}(1+pq)$ (1 mark)

iii) $x = a(p+q)$

$\therefore \frac{x}{a} = p+q$

$\therefore (\frac{x}{a})^2 = (p+q)^2 = p^2 + 2pq + q^2$

$= p^2 - 2pq + q^2 + 4pq$

$= (p-q)^2 + 4pq$

$= [\sqrt{3}(1+pq)]^2 + 4pq$

$= 3(1 + 2pq + p^2q^2) + 4pq$

$= 3 + 6pq + 3p^2q^2 + 10pq$

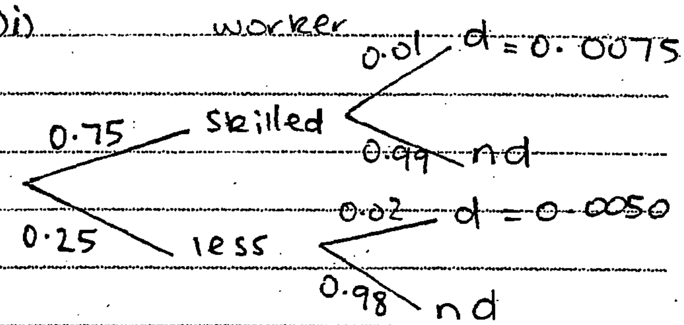
$\therefore \frac{x^2}{a^2} = 3 + 10pq + 3p^2q^2$

but $y = apq$, $\therefore pq = \frac{y}{a}$

$\therefore \frac{x^2}{a^2} = 3 + \frac{10y}{a} + \frac{3y^2}{a^2}$

$\therefore x^2 = 3a^2 + 10ay + 3y^2$ (2 marks)

c)i)



$\therefore$ probability (defective) $= (0.75 \times 0.01) + (0.25 \times 0.02) = 0.0125$ (1 mark)

ii) Probability (defective) = 0.0125

Probability (non-defective) $= 1 - 0.0125 = 0.9875$

By using binomial probability;

No more than one pair defective is none defective or 1 defective.

$\therefore P = {}^{20}C_0(0.9875)^{20} + {}^{20}C_1(0.0125 \times 0.9875^{19})$

$= 1 \times 0.7775... + (20 \times 0.0125 \times 0.7874...)$

$= 0.9744...$ (2 marks)

Question 5:

a) $g(x)=\frac{x}{1+x^2}$

i) domain: all real x

asymptotic behaviour: $x\to\pm\infty$

$y\simeq\frac{1}{x}\to 0$

∴ $y=0$ is a horizontal asymptote.

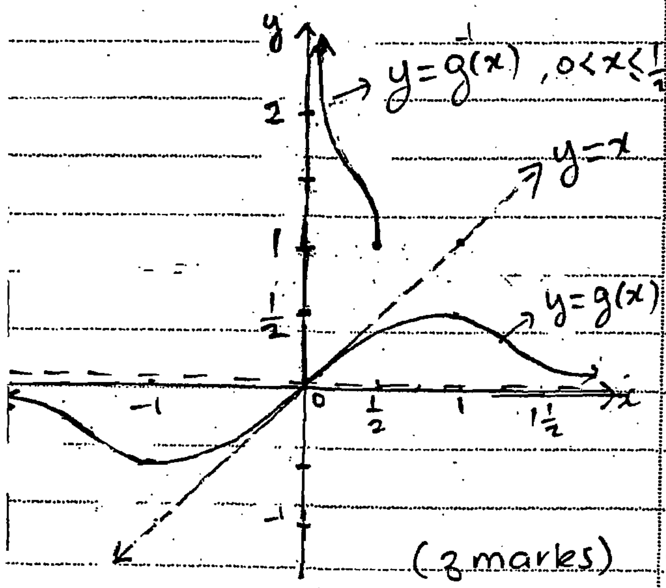
derivative: $\frac{dy}{dx}=\frac{1+x^2-2x^2}{(1+x^2)^2}$

$\frac{dy}{dx}=\frac{1-x^2}{(1+x^2)^2}$

Let $\frac{dy}{dx}=0$ to find possible stationary turning points.

$1-x^2=0$ ∴ $x^2=1$ ∴ $x=\pm1$

x	-1.5	-1	-0.5	1	1.5
$\frac{dy}{dx}$	$-0.118\ldots$ $-$	0	0.48 $+$	0	$-0.118\ldots$
y	↘	min $(-1,-\frac{1}{2})$	↗	max $(1,\frac{1}{2})$	↘



(3 marks)

ii) $g(x)$ has an inverse function when it is a one-one function. That is, for every x-value there is one y-value and for every y-value there is one x-value.

The largest domain containing $x=2$ for which this occurs is $x\geq1$. (1 mark)

iii) On graph. (1 mark)

iv) Domain of function: $x\geq1$

Range of function: $0<y\leq\frac{1}{2}$

Domain of inverse function: $0<x\leq\frac{1}{2}$

Range of inverse function: $y\geq1$

By interchanging x and y values, we get,

$x=\frac{y}{1+y^2}$, $x(1+y^2)=y$, $y=x+xy^2$

$xy^2-y+x=0$, using the quadratic formula,

$y=\frac{1\pm\sqrt{1-4x^2}}{2x}$

Since the range is $y\geq1$, y is only possible when $y=\frac{1+\sqrt{1-4x^2}}{2x}$ (1 mark)

b) i) $x=2+4\cos^2\left(\frac{\pi}{2}t\right)$

but $\cos\pi t=2\cos^2\frac{\pi}{2}t-1$

∴ $2\cos\pi t=4\cos^2\frac{\pi}{2}t-2$

and $2+2\cos\pi t=4\cos^2\frac{\pi}{2}t$

∴ $x=2+(2+2\cos\pi t)$

$=4+2\cos\pi t$ ①

so $\dot{x}=-2\pi\sin\pi t$ ②

$\ddot{x}=-2\pi^2\cos\pi t$ ③

$=-\pi^2(2\cos\pi t)$

From ①, $2\cos\pi t=x-4$

∴ $\ddot{x}=-\pi^2(x-4)$ Period $=2$

Since the acceleration is proportional to the displacement, where the constant of its proportion is negative. Hence, the motion is simple harmonic about $x=4$. (2 marks)

ii) $\dot{x}=-2\pi\sin\pi t$. As maximum value

$\sin \pi t = 1$ then max velocity

2π cm/sec. (1 mark)

$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

$\times 2 + 2\times 3 + 3\times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

p 1: For $n=1$,

$S = 1\times 2 = 2$

$HS = \frac{1(2)(3)}{3} = \frac{6}{3} = 2$

$HS = RHS$ $\therefore$ Statement true for $n=1$

p 2: Assuming the statement

e for $n=k$.

$+2\times 3 + 3\times 4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3}$

aim is to prove the statement

e for $n=k+1$.

$+2\times 3 + \dots + k(k+1) + (k+1)(k+2)$

$\frac{(k+1)(k+2)(k+3)}{3}$

$= 1\times 2 + 2\times 3 + \dots + k(k+1) + (k+1)(k+2)$

$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

$(k+1)(k+2)\left(\frac{k}{3} + 1\right)$

$(k+1)(k+2)\left(\frac{k+3}{3}\right)$

$\frac{(k+1)(k+2)(k+3)}{3} = RHS$

ce, if the statement is true

$n=k$, it is also true for

$k+1$.

3: From step 1, statement

ue for $n=1$. Hence by step 2

ement must be true for $n=2$,

hence true for $n=3$, and

ce for all positive integers.

(3 marks)

Question 6:

a) $f(x) = \cos^{-1}\left(\frac{1-x}{1+x}\right) - 2\tan^{-1}\sqrt{x}$

i) Let $u = \frac{1-x}{1+x}$ $\therefore y = \cos^{-1} u$

$\frac{du}{dx} = \frac{-1-x-(1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2}$

$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-\frac{(1-x)^2}{(1+x)^2}}} = \frac{-1}{\sqrt{\frac{(1+x)^2}{(1+x)^2} - \frac{(1-x)^2}{(1+x)^2}}}$

$= \frac{-1}{\sqrt{\frac{4x}{(1+x)^2}}} = \frac{-1}{\frac{\sqrt{4x}}{1+x}} = \frac{-(1+x)}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{-(1+x)}{2\sqrt{x}} \times \frac{-2}{(1+x)^2}$

$= \frac{1}{\sqrt{x}(1+x)}$

If $z = 2\tan^{-1}\sqrt{x}$, let $u = \sqrt{x}$ $\therefore z = 2\tan^{-1} u$

$\frac{du}{dx} = \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

$\frac{dz}{du} = 2 \times \frac{1}{1+u^2} = \frac{2}{1+x}$

$\therefore \frac{dz}{dx} = \frac{dz}{du} \times \frac{du}{dx} = \frac{2}{1+x} \times \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}(1+x)}$

$\therefore f'(x) = \frac{1}{\sqrt{x}(1+x)} - \frac{1}{\sqrt{x}(1+x)}$

$\therefore f'(x) = 0$ (3 marks)

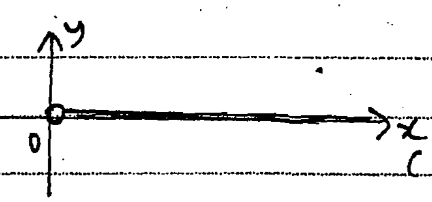
ii) As $f(x)$ is a continuous function on the domain $x > 0$, and $f'(x) = 0$.

$\therefore f(x) = C$. when $x=1$, $f(x) = \cos^{-1} 0 - 2\tan^{-1} 1$

so $f(x) = \frac{\pi}{2} - 2\times\frac{\pi}{4} = 0$

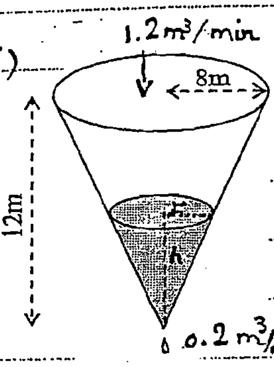
$\therefore f(x) = \frac{\pi}{2} - 2\times\frac{\pi}{4} = 0$

$\therefore f(x) = 0$ for $x > 0$



(1 mark)

b)i)



Let the water in the tank at a time t be in the shape of a cone of radius r and height h. Now, ΔABC is similar to ΔADE (equiangular). Volume of a cone at a time t is $V=\frac{1}{3}\pi r^2 t$.

By using the ratio of sides of similar Δ's $\frac{r}{8}=\frac{h}{12}$, $r=\frac{8h}{12}$

$\therefore r=\frac{2h}{3}$.

$\therefore V=\frac{1}{3}\pi h\left(\frac{4h^2}{9}\right)$, $V=\frac{4\pi}{27}h^3$

at $t=0$, $h=3$ m $\therefore V=4\pi\ m^3$.

As water enters the tank at a rate of $1.2\ m^3/min$, and leaks from the tank at a rate rate of $0.2\ m^3/min$, then $\frac{dV}{dt}=1.2-0.2$

$=1\ m^3/min$ $\therefore V=\int 1\,dt = t+c$

But when $t=0$, $V=4\pi$.

$\therefore V=(t+4\pi)\ m^3/min$. (2 marks)

ii) $\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$, but $V=\frac{4\pi}{27}h^3$ (part i)

$\therefore \frac{dV}{dh}=\frac{4\pi}{27}\times 3h^2=\frac{4\pi}{9}h^2$

$\frac{dV}{dt}=1$

$\therefore 1=\frac{4\pi}{9}h^2\cdot\frac{dh}{dt}$ $\therefore \frac{dh}{dt}=\frac{9}{4\pi h^2}$

Now $t=28\pi$, $V=28\pi+4\pi=32\pi$

$\frac{4\pi}{27}h^3=32\pi$ $h^3=216$, $\therefore h=6$

$\therefore \frac{dh}{dt}=\frac{9}{4\pi\times 6^2}$ $\therefore \frac{dh}{dt}=\frac{1}{16\pi}\ m/min$

(3 marks)

iii) Area of water in contact with the wall is $A=\pi rs$ (s is slant height)

but $s=\sqrt{h^2+r^2}$ (pythagoras' theorem)

$=\sqrt{h^2+\left(\frac{2h}{3}\right)^2}$

$=\sqrt{h^2+\frac{4h^2}{9}}$

$=\sqrt{\frac{13h^2}{9}}=\frac{h}{3}\sqrt{13}$ $(h>0)$

$A=\pi rs=\pi\times\frac{2h}{3}\times\frac{\sqrt{13}}{3}h=\frac{2\sqrt{13}}{9}\pi h^2$

$\frac{dA}{dh}=\frac{4\sqrt{13}}{9}\pi h$

$\frac{dA}{dt}=\frac{dA}{dh}\cdot\frac{dh}{dt}=\frac{4\sqrt{13}}{9}\pi h\times\frac{9}{4\pi h^2}=\frac{\sqrt{13}}{h}$

When $t=112\pi$ $\therefore V=104\pi+4\pi=108\pi$

$\therefore 108\pi=\frac{4\pi}{27}h^3$, $h^3=729$, $h=9m$

$\therefore \frac{dA}{dt}=\frac{\sqrt{13}}{9}\ m^2/min$ (3 marks)

Question 7:

a) LHS $=\left(1+\frac{1}{x}\right)^n(1+x)^n=(1+x^{-1})^n(1+x)^n$

$=({}^nC_0+{}^nC_1x^{-1}+\dots+{}^nC_{n-3}x^{3-n}+{}^nC_{n-2}x^{2-n}+{}^nC_{n-1}x^{1-n}+{}^nC_nx)$
$({}^nC_0+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+{}^nC_4x^4+\dots+{}^nC_nx^n)$.

$C_0C_3+C_1C_4+\dots+C_{n-3}C_n$ would represent the co-efficient of x^3 in the expansion of LHS.

To get x^3 in RHS we need to get the coefficient of x^{n+3} in $(1+x)^{2n}$. T_{r+1} in $(1+x)^{2n}$ is ${}^{2n}C_rx^r$ $\therefore$ coef. of x^{n+3} is obtained for $r=n+3$ $\therefore$ coefficient required is ${}^{2n}C_{n+3}$, but ${}^{2n}C_{n+3}=\frac{(2n)!}{(n+3)!(2n-(n+3))!}$

$=\frac{2n!}{(n+3)!(n-3)!}$

Hence, by equating the coefficient of x^3 on each side of the identity:

$\therefore C_0C_3+C_1C_2+\dots+C_{n-3}C_n=\frac{(2n)!}{(n+3)!(n-3)!}$

(3 marks)

b) For the target at B, there are two motions: horizontal motion and vertical motion.

rizontal Motion:

$\ddot{x} = 0$, by integrating with respect

time t, $\dot{x} = C_1$. When $t=0$, $\dot{x}=0$

$C_1 = 0$

$\dot{x} = 0$

y integrating with respect of

me t, $x = C_2$. When $t=0$, $x=d$

$\boxed{\therefore x = d.}$

ertical motion:

$\ddot{y} = -g$. By integrating with

spect of time t, $\dot{y} = -gt + K_1$

hen $t=0$, $\dot{y} = 0$ $\therefore K_1 = 0$

$\dot{y} = -gt$

y integrating with respect

f time t $\therefore y = -\frac{1}{2} g t^2 + K_2$

hen $t=0$, $y=d$ $\therefore K_2 = d$

$\boxed{y = -\frac{1}{2} g t^2 + d}$ (1 mark)

The initial speed of V is

irected towards (d,d). Let

be the angle of v with

ositive x-axis.

$\tan\alpha = \frac{d}{d} = 1$ $\therefore \alpha = 45^\circ$

milarily, the angle of U with

gative x axis is 45°.

The angle of U with positive

axis is $180^\circ - 45^\circ = 135^\circ$.

rizontal motion of the projectile

om O.

$\ddot{x} = 0$ By integrating with respect

time t, $\dot{x} = C_1$. when $t=0$,

$\dot{x} = V\cos 45^\circ = \frac{V}{\sqrt{2}}$ $\therefore C_1 = \frac{V}{\sqrt{2}}$ $\therefore \dot{x} = \frac{V}{\sqrt{2}}$

By integrating with respect of time t,

$x = \frac{V}{\sqrt{2}} t + C_2$. When $t=0$, $x=0$

$\therefore C_2 = 0$ $\therefore x = \frac{V}{\sqrt{2}} t + C_2$ When $t=0$, $x=0$

$\therefore C_2 = 0$ $\boxed{\therefore x = \frac{V}{\sqrt{2}} t}$

Vertical motion of the projectile from O.

$\ddot{y} = -g$. By integrating with respect of

time t. $\dot{y} = -gt + K_1$. When $t=0$,

$\dot{y} = V\sin 45^\circ = \frac{V}{\sqrt{2}}$ $\therefore K_1 = \frac{V}{\sqrt{2}}$

$\therefore \dot{y} = -gt + \frac{V}{\sqrt{2}}$

By integrating with respect of time t.

$y = -\frac{1}{2} g t^2 + \frac{V}{\sqrt{2}} t + K_2$. When $t=0$, $y=0$.

$\therefore K_2 = 0$ $\boxed{\therefore y = -\frac{1}{2} g t^2 + \frac{V}{\sqrt{2}} t}$

Horizontal motion of the projectile from A.

$\ddot{x} = 0$. By integrating with respect of

time t, $\dot{x} = C_1$. when $t=0$, $\dot{x} = U\cos 135$

$= \frac{-U}{\sqrt{2}}$ $\therefore C_1 = \frac{-U}{\sqrt{2}}$ $\therefore \dot{x} = \frac{-U}{\sqrt{2}}$

By integrating with respect of time t,

$x = \frac{-U}{\sqrt{2}} t + C_2$. when $t=0$, $x = 2d$

$\therefore C_2 = 2d$ $\boxed{x = \frac{-U}{\sqrt{2}} t + 2d}$

Verticle motion of the projectile from A.

$\ddot{y} = -g$. By integrating with respect of

time t. $\dot{y} = -gt + K_1$, when $t=0$,

$\dot{y} = U\sin 135^\circ = \frac{U}{\sqrt{2}}$ $\therefore K_1 = \frac{U}{\sqrt{2}}$ $\therefore \dot{y} = -gt + \frac{U}{\sqrt{2}}$

By integrating with respect of time t.

$y = -\frac{1}{2} g t^2 + \frac{U}{\sqrt{2}} t + K_2$. When $t=0$, $y=0$

$\therefore K_2 = 0$ $\therefore \boxed{y = -\frac{1}{2} g t^2 + \frac{U}{\sqrt{2}} t}$ (3 marks)

iii) In order for the projectile to hit the

target, they have to have the same

coordinates at the same time. The equations

of the motions of the projectile from O

are $x = \frac{V}{\sqrt{2}} t$ and $\boxed{y = -\frac{1}{2} g t^2 + \frac{V}{\sqrt{2}} t}$, and the equations of the motions of the target are $x=d$ and $y=-\frac{1}{2}gt^2+d$.

let $y=y$

$\therefore -\frac{1}{2}gt^2+\frac{V}{\sqrt{2}}t=-\frac{1}{2}gt^2+d$

$t=\frac{\sqrt{2}d}{V}$. when $t=\frac{\sqrt{2}d}{V}$,

$x_{target}=d$ and $x_{projectile}$ from O is $x=\frac{V}{\sqrt{2}}t=\frac{V}{\sqrt{2}}\times\frac{\sqrt{2}d}{V}=d$

$\therefore$ At $\boxed{t=\frac{\sqrt{2}d}{V}}$ the projectile from O hits the target, as they have the same co ordinates.

But in order for the impact to occur in the air it should be $y>0$ for $t=\frac{\sqrt{2}d}{V}$. But $y=-\frac{1}{2}gt^2+d$

$\therefore -\frac{1}{2}g\left(\frac{\sqrt{2}d}{V}\right)^2+d>0$

$-\frac{1}{2}\times\frac{2d^2}{V^2}+d>0$ $\quad -\frac{gd^2}{V^2}+d>0$

(dividing by $d>0$) $-\frac{gd}{V^2}+1>0$

$1>\frac{gd}{V^2}$ $\therefore V^2>gd$ $\therefore V>\sqrt{gd}$ (as $V>0$).

$\therefore$ The impact occurs in the air $V>\sqrt{gd}$. (3 marks)

iv) After the impact, the fragment falls vertically down.

$\therefore$ The equations of the motion of the target can be used for the motion of the fragment. Now the projectile from A hits the fragment from the same co ordinate at the same time.

For $y_{projectile}=y_{fragment}$

$-\frac{1}{2}gt^2+\frac{U}{\sqrt{2}}t=-\frac{1}{2}gt^2+d$

$t=\frac{\sqrt{2}d}{U}$

this time is the time taken for the projectile from A to hit the fragment.

The time taken for the projectile from O to hit the target is $t=\frac{\sqrt{2}d}{V}$.

$\therefore T=\text{time}_{fragment}-\text{time}_{target}$

$=\frac{\sqrt{2}d}{U}-\frac{\sqrt{2}d}{V}$

$=\sqrt{2}d\left(\frac{1}{U}-\frac{1}{V}\right)$ (2 marks)