Caringbah High School

2013

Trial HSC Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
(Black pen is preferred)
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2 – 4

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 5 – 8

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
For Questions 1–10, use the multiple-choice answer sheet on page 9. Please detach this from the exam paper and submit with your answer booklets.

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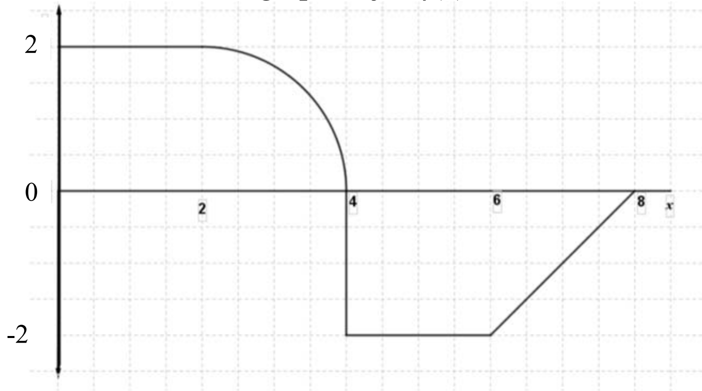
1 This is the graph of $y = f(x)$, $0 \le x \le 8$.



The value of $\int_0^8 f(x)dx$ is

A) $\frac{\pi}{4} - 2$ B) $\pi - 2$

C) $\frac{\pi}{4} + 10$ D) $10 + \pi$

2 Let $f(x) = 3cos^{-1}\left(\frac{x}{2}\right)$.
The domain of the function *f(x)* is given by

A) $-\frac{1}{3} \le x \le \frac{1}{3}$ B) $-\frac{1}{2} \le x \le \frac{1}{2}$

C) $-2 \le x \le 2$ D) $-3 \le x \le 3$

3 The point (3, -4) divides the interval AB externally in the ratio 3:2. If the coordinates of A are (6,5), then the coordinates of B are

A) (5, 2) B) (8, 11)

C) (4, -1) D) (1, -10)

4

$$\frac{d(tan^{-1}3x)}{dx} =$$

A) $\frac{3}{9+\ x^2}$

B) $\frac{1}{9+\ x^2}$

C) $\frac{1}{1+\ 9x^2}$

D) $\frac{3}{1+\ 9x^2}$

5 The variable point P $(5t, t^2)$ lies on a parabola. The Cartesian equation for this parabola is

A) $y = \frac{x^2}{4}$

B) $x^2 = 10y$

C) $y = 25x^2$

D) $x^2 = 25y$

6 α, β and γ are roots of the equation $x^3 - 3x^2 + 1 = 0$.

The value of $\alpha\beta + \alpha\gamma + \beta\gamma$ is

A) -1

B) 0

C) 1

D) 3

7 A particle undergoes SHM about the origin. Its displacement in cm is given by

$$x = 3\cos\left(2t + \frac{\pi}{3}\right).$$

The particle is at rest when

A) $x = -3$

B) $x = 0$

C) $t = \frac{\pi}{6}$

D) $t = 0$

8

$$\int \frac{-dx}{\sqrt{9-\ x^2}}$$

Which of the following may be a solution?

A) $\cos^{-1}\frac{x}{3}$

B) $\sin^{-1}\frac{x}{3}$

C) $\cos^{-1} 3x$

D) $\sin^{-1} 3x$

9 The vertical asymptote on the graph of $y = \dfrac{3x}{x-2}$ is

A) $x = 2$

B) $y = 0$

C) $x = 3$

D) $y = 3$

10 Given that $(2x - 3)$ is a factor of $P(x) = 2x^3 - 3\text{x} + \text{c}$, the value of c is

A) -45

B) $-2\,\frac{1}{4}$

C) $2\,\frac{1}{4}$

D) 45

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Evaluate $\lim_{x\to\infty} \dfrac{3x+2}{5-x}$ **1**

b) Differentiate $\sin^{-1}(x^2)$, with respect to x. **2**

c) Sketch the graph of $y = 3\sin\left(\frac{x}{2}\right)$ in the domain $0 \le x \le 2\pi$. **2**

d) Using the substitution $u = 4 - x^3$, **3**

$$\text{Evaluate} \int_{-1}^{1} x^2\sqrt{4-x^3}\,dx$$

e) The line $y = 2x - 3$ intersects with the curve $y = 2x^3 - 15$ at the point $(2,1)$. Find the size of the angle between the line and the curve at the point of intersection. (Answer to nearest degree) **3**

f) Find all values for x that satisfy $\dfrac{5}{x-4} \le x$ **2**

g) The function $f(x) = x^2 - e^x$ has a root near $x = 3$. Use one application of Newton's method to find a better approximation. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) Evaluate $\lim_{x \to 0} \dfrac{3x}{\sin 2x}$ **1**

b) Find the general solution to **2**

$$\sqrt{3}\ \tan x - 1 = 0$$

Answer as an exact value in radians.

c) A pendulum swings freely due to gravity and is friction free. When viewed from above, the end of the pendulum executes simple harmonic motion, with a period of π seconds and an amplitude of 1·2 m.

i) Explain why the acceleration, $\ddot{x}$, of the pendulum is given by $\ddot{x} = -4x$, where x is the position at any time, t. **1**

ii) Using part i), show that the maximum velocity of the end of the pendulum is $2 \cdot 4\ ms^{-1}$. **2**

d) i) $x^2 + 8x + 20$ can be expressed in the form $(x + a)^2 + b^2$. **2**

Find values for a and b.

ii) Hence or otherwise find **2**

$$\int \frac{1}{x^2 + 8x + 20}\, dx$$

e) A spherical beach ball is being inflated at a rate of 12 mm^3 per second. **2**

Calculate the rate that the radius is increasing when the surface area is 5 000 mm^2. (NB. $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$)

f) The half-life of a substance is the time taken for half of the substance to decay. The carbon isotope ^{14}C decays at a rate proportional to it mass. It has been shown that ^{14}C has a half-life of 5580 years. **3**

A fossil that was tested contained 40% of the ^{14}C it would have originally contained.

Estimate the age of the fossil.

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) Solve, $\cos 2x = \sqrt{3}\cos x - 1,$ for $0 \le \theta \le \frac{\pi}{2}.$ **2**

b) i) Differentiate $x\sin^{-1}(x) + \sqrt{1-x^2}$ **2**

ii) Hence evaluate $\int_0^1 \sin^{-1} x\, dx$ **2**

c) The 2 points P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$, with $a > 0$.

The chord PQ passes through the focus of the parabola.

i) Show that $pq = -1$. **1**

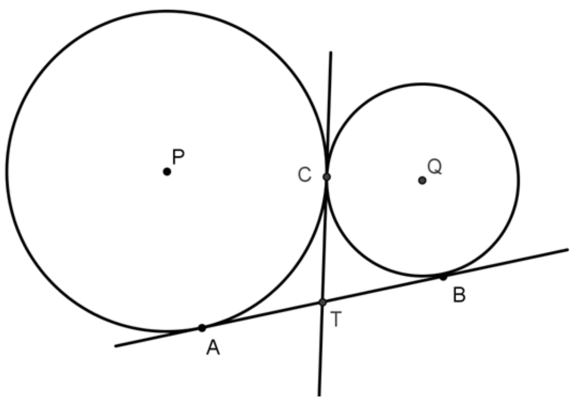
ii) The tangent at P has the equation $y = px - ap^2$ **2**

The tangents from P and Q intersect at T. Show that T lies on the directrix, $y = -a$.

d) Find all values of θ, $0 \le \theta \le \pi$ such that **3**

$\sqrt{2}\sin\theta + \cos\theta = 1$

e) Two circles with centres P and Q touch externally at C and have a common tangent that touches at A and B, as shown. The common tangent at C meets AB in T.



i) Show that T is the midpoint of AB. **1**

ii) Show that C, T, A and P are concyclic. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) Find $\int \sin^2 x \, dx$ **2**

b) Let $f(x) = \dfrac{1}{1 + x^3}$ for all x. **1**

Find an expression for the inverse function $f^{-1}(x)$, in terms of x.

c) i) Sketch the curve $y = \cos^{-1} x$ **1**

ii) The area between the curve $y = \cos^{-1} x$, the line $x = -1$ and the x-axis is rotated about the x-axis. Use Simpson's rule with 5 function values to approximate the volume of the solid formed. **3**

d) Prove by mathematical induction that **3**

$$7^{2n} - 3^{3n} \text{ is divisible by } 11, \text{for all integers } n \geq 1.$$

e) If an archer fires an arrow with a velocity of $50\ ms^{-1}$ at an angle of θ to the horizontal, it can be shown that the equations of motion are given by

$$x = 50t\cos\theta \quad \text{and} \quad y = \frac{-gt^2}{2} + 50t\sin\theta \quad (\text{do not prove this}).$$

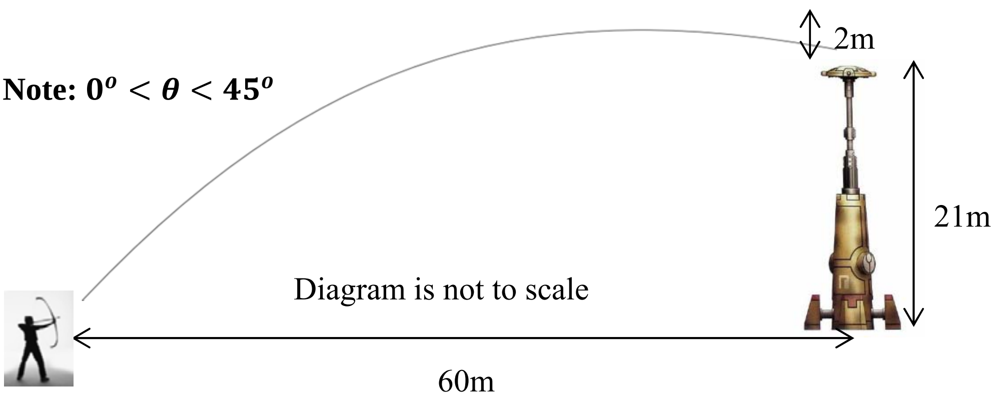
i) Show the Cartesian equation for the flight of the arrow is given by **2**

$$y = x\tan\theta - \frac{gx^2}{5000}\sec^2\theta$$

ii) In the 1992 Olympic Games in Barcelona, paralympian Antonio Rebello lit the Olympic cauldron in a most unique manner. From a horizontal distance of 60 metres from the base of the cauldron he fired a lit arrow across the top of the cauldron. The top of the cauldron was 21 metres higher than him. He had to shoot the arrow to within 2 metres above the cauldron to ignite the rising gas.

Using $g = 10$, find the range of angles from the horizontal that Antonio Rebello could aim through to successfully light the Olympic flame. **3**

Note: $0^o < \theta < 45^o$



End of Examination

Name: ______________________________

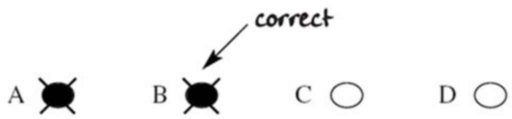
Multiple Choice Answer Sheet

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.



1)	A ○	B ○	C ○	D ○
2)	A ○	B ○	C ○	D ○
3)	A ○	B ○	C ○	D ○
4)	A ○	B ○	C ○	D ○
5)	A ○	B ○	C ○	D ○
6)	A ○	B ○	C ○	D ○
7)	A ○	B ○	C ○	D ○
8)	A ○	B ○	C ○	D ○
9)	A ○	B ○	C ○	D ○
10)	A ○	B ○	C ○	D ○

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Section 1

1) B 2) C 3) A 4) D 5) D
6) B 7) A 8) A 9) A 10) C

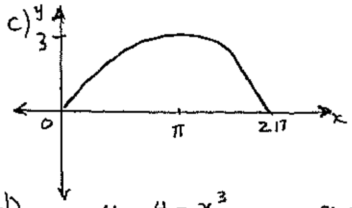
Section 2

uestion 11

a) -3

b) $\dfrac{2x}{\sqrt{1-x^4}}$

c)



d) $u = 4 - x^3$ $\quad x=1 \quad u=3$

$du = -3x^2 dx$ $\quad x=-1 \quad u=5$

$$\int_{-1}^{1} x^2\sqrt{4-x^3}\,dx = \int_5^3 \sqrt{u}\cdot\frac{-du}{3}$$
$$= -\frac{1}{3}\int_5^3 u^{\frac{1}{2}}\,du$$
$$= -\frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_5^3$$
$$= -\frac{2}{9}\left[\sqrt{3^3}-\sqrt{5^3}\right]$$
$$= \frac{2}{9}\left[5\sqrt{5}-3\sqrt{3}\right]$$

e) $y = 2x^3 - 15$

$y' = 6x^2$

$x = 2 \quad m_1 = 24 \qquad y = 2x - 3 \quad m_2 = 2$

$$\tan\theta = \frac{24-2}{1+24\times 2}$$
$$= \frac{22}{49}$$
$$\therefore \theta = 24^\circ$$

f) critical values $\quad$ test $x=-2$ F, $x=0$ T, $x=4\frac{1}{2}$ F, $x=6$ T

$x=4 \qquad x^2-4x=5$

$x=5,\ x=-1$

$\therefore -1 \le x < 4$ or $x \ge 5$

g) $f(x) = x^2 - e^x \qquad f(3) = 9 - e^3$

$f'(x) = 2x - e^x \qquad f'(3) = 6 - e^3$

$$x_2 = 3 - \frac{(9-e^3)}{(6-e^3)}$$
$$\doteqdot 2.21 \quad (3 \text{ sig figs})$$

Question 12

a) $\frac{3}{2}$

b) $\tan x = \frac{1}{\sqrt{3}}$ $\quad$ acute $x = \frac{\pi}{6}$

$x = n\pi + \frac{\pi}{6} \qquad n = 0, \pm1, \pm2, \ldots$

c) i) $\ddot{x} = -n^2x$ defines SHM. $T = \frac{2\pi}{n}$

$T = \pi$ then $n=2$ so $\ddot{x} = -4x$.

ii) $v^2 = n^2(a^2 - x^2) \qquad n=2,\ a=1.2$

max v when $x=0$ $\therefore V = na$

$= 2.4\ ms^{-1}$

d) i) $x^2 + 8x + 20 = x^2 + 8x + 16 + 4$

$= (x+4)^2 + 4$

$a=4 \quad b=\pm2$

ii) $\displaystyle\int \frac{1}{(x+4)^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right) + c$

e) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2 \qquad \frac{dV}{dt} = 12\ mm^3/s$

$$\frac{dr}{dt} = \frac{dr}{dV}\times\frac{dV}{dt}$$
$$= \frac{1}{4\pi r^2}\times 12 \qquad \text{but } 4\pi r^2 = 5000$$
$$= \frac{12}{5000}$$
$$= \frac{3}{1250}\ mm/s \qquad (2.4\times10^{-3}\ mm\,s^{-1})$$

f) $M = M_0 e^{-kt}$

$\frac{1}{2}M_0 = M_0 e^{-5580k}$

$$k = \frac{-\ln(\frac{1}{2})}{5580} \qquad (k \doteqdot 1.2422)$$

$0.4M_0 = M_0 e^{-kt}$

$$t = \frac{-\ln(0.4)}{k}$$
$$= 7376 \text{ years}$$

Question 13

a) $\cos 2x = 2\cos^2 x - 1$

so $2\cos^2 x - 1 = \sqrt{3}\cos x - 1$

$2\cos^2 x - \sqrt{3}\cos x = 0$

$\cos x(2\cos x - \sqrt{3}) = 0$

$\cos x = 0$ or $\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$

b) i) let $y = x\sin^{-1}x + (1-x^2)^{\frac{1}{2}}$

$y' = x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x$

$= \sin^{-1}x$

ii) $\displaystyle\int_0^1 \sin^{-1}x\,dx = \left[x\sin^{-1}x + \sqrt{(1-x^2)}\right]_0^1$

$= \left[(\sin^{-1}(1) + \sqrt{1-1}) - 0 + \sqrt{1-0}\right]$

$= \frac{\pi}{2} - 1$

c) focus $S(0, a)$

need $m_{PS} = m_{QS}$ $\quad \dfrac{ap^2-a}{2ap} = \dfrac{aq^2-a}{2aq}$

$$\frac{p^2-1}{p} = \frac{q^2-1}{q}$$

$p^2q - q = pq^2 - p$

$p^2q - pq^2 = q - p$

$pq(p-q) = -(p-q)$

$pq = -1$

ii) $y = px - ap^2 \qquad y = qx - aq^2$

$x = \dfrac{y+ap^2}{p} \qquad x = \dfrac{y+aq^2}{q}$

$$\therefore \frac{y+ap^2}{p} = \frac{y+aq^2}{q}$$

$qy + ap^2q = py + apq^2$ $\quad$ as $pq=$

$qy - ap = py - qa$

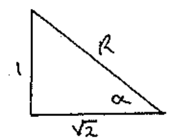
$qy - py = ap - aq$

$y(q-p) = -a(q-p)$

$y = -a$

d) $\sqrt{2}\sin\theta + \cos\theta = R\sin(\theta + \alpha)$

$R\cos\alpha = \sqrt{2} \qquad R\sin\theta = 1$



$R = \sqrt{3}$

$\tan\alpha = \frac{1}{\sqrt{2}}$

$\alpha = 0.615$

so $\sqrt{3}\sin(\theta + 0.615) = 1$

$\theta + 0.615 = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = 0.615 - 0.615$ or $\theta = (\pi - 0.615) - 0.$

as $0 \le \theta \le 2\pi \qquad \theta = 0,\ 1.911$

e) i) $AT = CT$ and $BT = CT$ (tangents from poi

$\therefore AT = BT \Rightarrow T$ is midpoint of AB

ii) join PA and CA

$\angle PAT = \angle PCT = 90^\circ$ (radius ⊥ tangent)

$\therefore \angle PAT + \angle PCT = 180^\circ$

$\therefore C, T, A$ and P are concyclic. (opp angles supp.

Question 14

a) $\displaystyle\int \sin^2 x\,dx = \frac{1}{2}\int 1 - \cos 2x\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c$

$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$

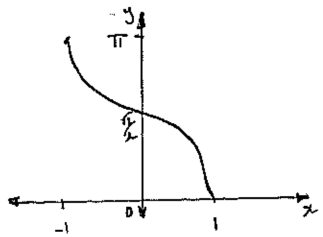
b) $y = \dfrac{1}{1+x^3}$ $\quad$ inverse $x = \dfrac{1}{1+y^3}$

$1 + y^3 = \frac{1}{x}$

$y^3 = \frac{1}{x} - 1$

$y = \sqrt[3]{\dfrac{1-x}{x}}$

c) i)



ii) $V = \pi \int_{-1}^{1} [\cos^{-1} x]^2 \, dx$

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
y^2	π^2	$\frac{4\pi^2}{9}$	$\frac{\pi^2}{4}$	$\frac{\pi^2}{9}$	0

$h = \frac{1}{2}$

$\frac{h}{3} = \frac{1}{6}$

$V \doteq \frac{\pi}{6}\left[\left(\pi^2 + \frac{\pi^2}{4} + 4 \times \frac{4\pi^2}{9}\right) + \left(\frac{\pi^2}{4} + 0 + 4 \times \frac{\pi^2}{9}\right)\right]$

$= \frac{67\pi^3}{108}$ (19.24)

d) $n = 1$ $\quad 7^2 - 3^3 = 22$ $\therefore$ true for $n = 1$

Assume true for $n = k$

$7^{2k} - 3^{3k} = 11M$

Prove true for $n = k+1$

$7^{2k+2} - 3^{3k+3} = 7^2\left[7^{2k} - 3^{3k}\right] + 7^2 \times 3^{3k} - 3^{3k+3}$

$= 7^2 \times 11M + 3^{3k}(7^2 - 3^3)$

$= 11(49M + 2 \times 3^{3k})$

$\therefore$ If true for $n=k$, then also true for $n=k+1$.
As it is true for $n=1$ then by mathematical induction it is true for all $n = 1, 2, 3, \ldots$

e) $x = 50t\cos\theta$

$t = \frac{x}{50\cos\theta}$

$y = -\frac{g}{2}\left(\frac{x}{50\cos\theta}\right)^2 + 50\left(\frac{x}{50\cos\theta}\right)\sin\theta$

$= \frac{-gx^2}{5000}\sec^2\theta + x\tan\theta$

ii) When $x = 60$ $\quad 21 \le y \le 23$

$23 = 60\tan\theta - \frac{10 \times 60^2}{5000}\left(1 + \tan^2\theta\right)$

$\frac{36}{5}\tan^2\theta - 60\tan\theta + \left(23 + \frac{36}{5}\right) = 0$

$A = \frac{36}{5}$ $\quad B = -60$ $\quad C = \left(23 + \frac{36}{5}\right)$

$\tan\theta = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

as $\theta < 45$

$\tan\theta = 0.5381$

$\theta = 28.3°$ $\quad (28° 17')$

using $C = \left(21 + \frac{36}{5}\right)$

$\tan\theta = \frac{1}{2}$

$\theta = 26.6°$ $\quad (26° 34')$

$\therefore$ Angle must be between $26.6°$ and $28.3°$