CARINGBAH HIGH SCHOOL

2013

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–12

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 Which of the following are the foci for the ellipse

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

(A) $(\mathbf{0},\pm\frac{3\sqrt{7}}{4})$ (B) $(\pm\sqrt{\mathbf{7}},\mathbf{0})$

(C) $(\mathbf{0},\pm\sqrt{\mathbf{7}})$ (D) $(\frac{3\sqrt{7}}{4},0)$

2 Which is the correct answer to the following integral?

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} sin^7(x)cos^4(x)\;dx$$

(A) $\mathbf{2}\times\int_{0}^{\frac{\pi}{4}} \boldsymbol{sin^7(x)cos^4(x)\;dx}$ (B) $\frac{\mathbf{1024-533\sqrt{2}}}{\mathbf{36960}}$

(C) $-\frac{\mathbf{1024-533\sqrt{2}}}{\mathbf{36960}}$ (D) zero

3 Let α,β and γ be the roots of the cubic equation $x^3-5x^2+13x-7=0$ .Which of the following is the equation with roots α^2,β^2 and γ^2 ?

(A) $\mathbf{7x^3-13x^2+5x-1=0}$ (B) $\mathbf{x^3+x^2+99x-49=0}$

(C) $\mathbf{x^3+5x^2-13x-7=0}$ (D) $\mathbf{49x^3+99x^2+x-1=0}$

4 Given that $x^2 + y^2 + xy = 12$, which of the following is true?

(A) $\frac{dy}{dx} = \frac{2x + y}{2y + x}$

(B) $\frac{dy}{dx} = -\frac{2x + y}{2y + x}$

(C) $\frac{dy}{dx} = \frac{2x - y}{2y + x}$

(D) $\frac{dy}{dx} = \frac{-2x + y}{2y + x}$

5 The equation $|z - 1 - 3i| + |z - 9 - 3i| = 10$ corresponds to an ellipse in the Argand diagram. Which of the following is the complex number corresponding to the centre of the ellipse?

(A) $5 + 3i$

(B) $-5 + 3i$

(C) $-5 - 3i$

(D) $5 - 3i$

6 The point $T(acos\theta, asin\theta)$ lies on the circle $x^2 + y^2 = r^2$. Which of the following gives the equation of the tangent at T?

(A) $xcos\theta + ysin\theta = a$

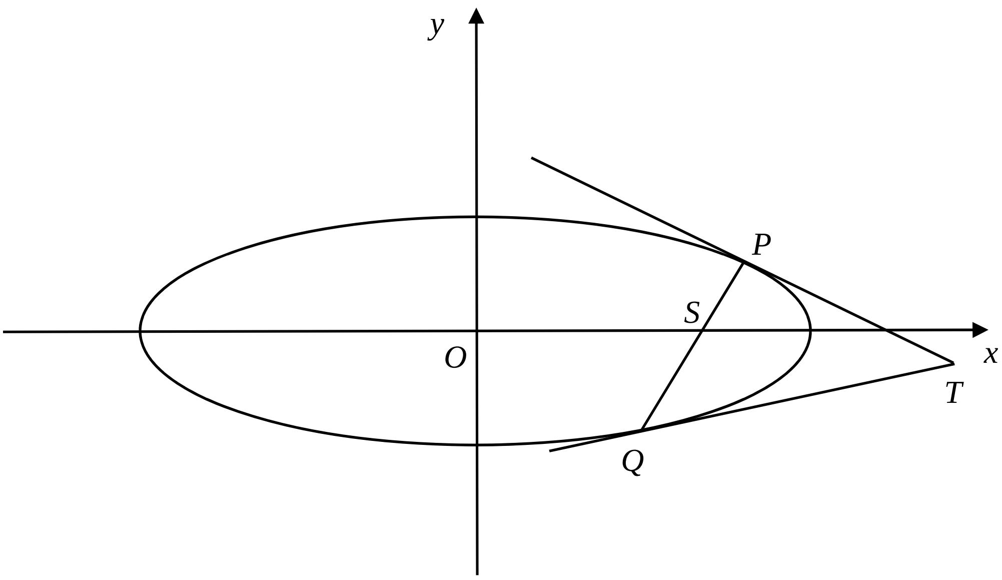
(B) $xcos\theta - ysin\theta = a$

(C) $xcos\theta - ysin\theta = a^2$

(D) $xcos\theta + ysin\theta = a^2$

7

The point P lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The chord through P and the focus $S(ae,0)$ meets the ellipse at Q. The tangents to the ellipse at P and Q meet at the point $T(x_0,y_0)$, so the equation of PQ is $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$. (Note that T lies on the directrix).



What is the value of the ratio $\frac{PS}{ST}$, given that this ratio is constant?

(A) $\boldsymbol{e^2}$

(B) ae

(C) $\boldsymbol{\frac{a}{e}}$

(D) $\boldsymbol{e}$

8 Suppose $\omega^3=1$, $\omega\neq 1$ and k is a positive integer.

What are the two values of $1+\omega^k+\omega^{2k}$?

(A) 3, 0

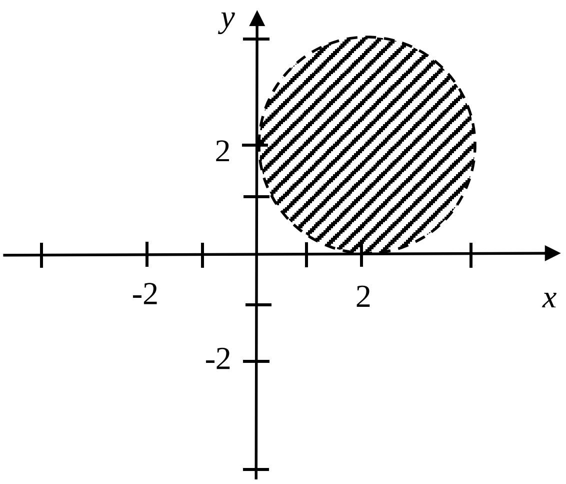
(B) 3, 1

(C) 1, 0

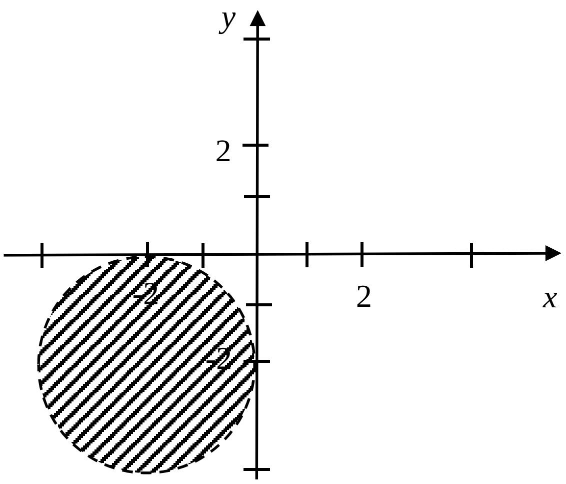
(D) None of the above

9 Which of the following is the graph of $|z - 1 - i| < 2$?

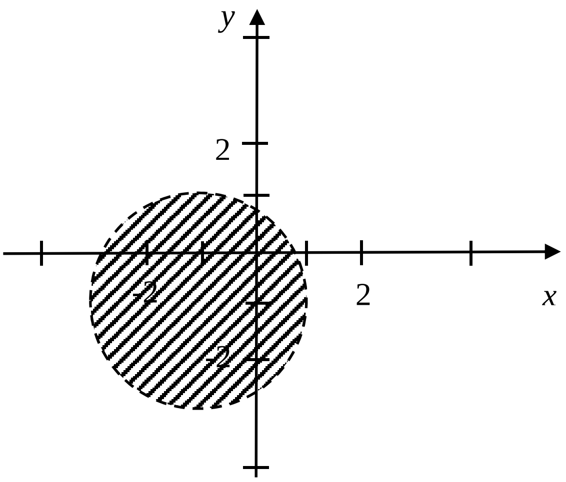
(A)



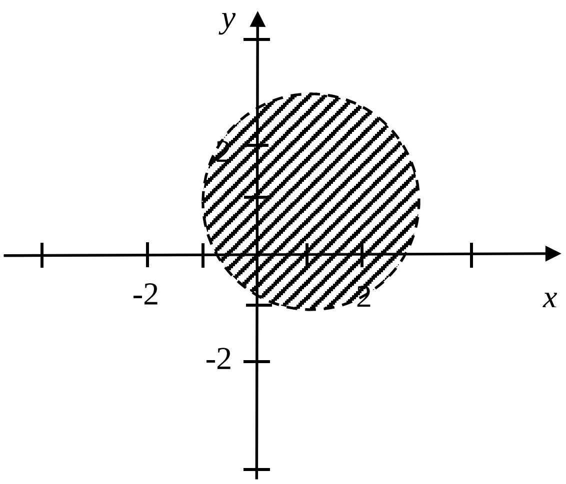
(B)



(C)



(D)



10 Given that $\cos(a+b)\,x + \cos(a-b)\,x = 2\cos(ax)\cos(bx)$, which of the following is the answer for

$$\int \cos(3x)\cos(2x)\,dx \ ?$$

(A) $\frac{1}{2}(\boldsymbol{cos5x + cosx}) + \boldsymbol{c}$

(B) $\frac{1}{10}\boldsymbol{cos5x} + \frac{1}{2}\boldsymbol{cosx} + \boldsymbol{c}$

(C) $\frac{1}{10}\boldsymbol{sin5x} + \frac{1}{2}\boldsymbol{sinx} + \boldsymbol{c}$

(D) $\frac{1}{2}(\boldsymbol{sin5x + sinx}) + \boldsymbol{c}$

END OF MULTIPLE CHOICE QUESTIONS

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available. In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW booklet. **Marks**

(a) Find

(i) $\int \frac{t^2 - 2}{t^3}\, dt$ **2**

(ii) $\int xe^x\, dx$ **2**

(iii) $\int \frac{2x}{(x+1)(x+3)}\, dx$ **3**

(b) By using the substitution $u = x - 4$ evaluate

$$\int_4^{4\cdot 5} \frac{dx}{\sqrt{(x-3)(5-x)}}$$ **3**

(c) (i) If **3**

$$u_n = \int_0^{\frac{\pi}{2}} x^n sinx\, dx, \qquad n \geq 2$$

prove that

$$u_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)u_{n-2}$$

(ii) Hence evaluate **2**

$$\int_0^{\frac{\pi}{2}} x^2 sinx\, dx$$

Question 12 (15 marks) Start a NEW booklet. **Marks**

(a) The complex number w is given by $w = -1 + i\sqrt{3}$.

(i) Show that $w^2 = 2\overline{w}$. **2**

(ii) Evaluate $|w|$ and $\arg w$. **2**

(iii) Show that w is a root of $w^3 - 8 = 0$ **1**

(b) Sketch the locus of z satisfying:

(i) $Re(z) = |z|$ **2**

(ii) Both $Re(z) \geq 2$ and $|z - 1| \leq 2$ **3**

(c) Given that a and b are real numbers and **2**

$$\frac{a}{1+i} + \frac{b}{1+2i} = 1$$

find the values of a and b.

(d) The complex numbers z_1, z_2, z_3 and z_4 are represented in the complex plane by the points A, B, C and D respectively. **3**

If $z_1 + z_3 = z_2 + z_4$ prove ABCD is a parallelogram.

Question 13 (15 marks) Start a NEW booklet. **Marks**

(a) The equation $x^3 + bx^2 + x + 2 = 0$, where *b* is a real number, has roots α, β, γ.

(i) Obtain an expression, in terms of *b*, for **2**

$$\alpha^2 + \beta^2 + \gamma^2$$

(ii) Hence determine the set of possible values of *b* if the roots of the above equation are all real. **1**

(iii) Write down the equation whose roots are **2**

$$2\alpha, 2\beta, 2\gamma$$

(b) Given that the polynomial $P(x) = 8x^4 - 36x^3 - 66x^2 - 35x - 6$ has a zero of multiplicity 3, find all the zeros of $P(x)$. **3**

(c) If *z* represents a complex number such that $z^5 = 1$, where $z \neq 1$.

(i) Deduce that **2**

$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$$

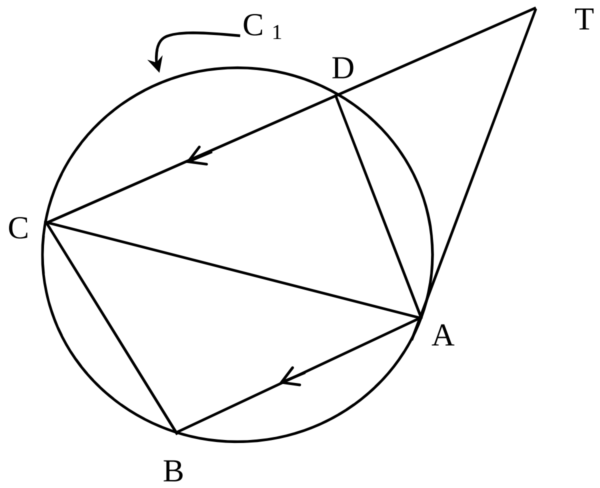
(ii) By substituting $x = z + \frac{1}{z}$ reduce the equation in (i) to a quadratic in *x*. **2**

(iii) Hence deduce that **3**

$$cos\frac{2\pi}{5}\ cos\frac{4\pi}{5} = -\frac{1}{4}$$

Question 14 (15 marks) Start a NEW booklet. **Marks**

(a) The points A, B, C and D lie on the circle C_1. From the exterior point T, a tangent is drawn to point A on C_1. The line CT passes through D and TC is parallel to AB.



(i) Copy or trace the diagram on to your page.

(ii) Prove that ΔADT is similar to ΔABC. **3**

The line BA is produced through A to point M, which lies on a second circle C_2. The points A, D, T also lie on C_2 and the line DM crosses AT at Q.

(iii) Show that ΔQMA is isosceles. **2**

(iv) Show that $TM = BC$. **2**

(b) (i) Prove that the normal to the hyperbola $xy = 4$ at the point $P(2p, \frac{2}{p})$ is given by **2**

$$p^3x - py = 2(p^4 - 1)$$

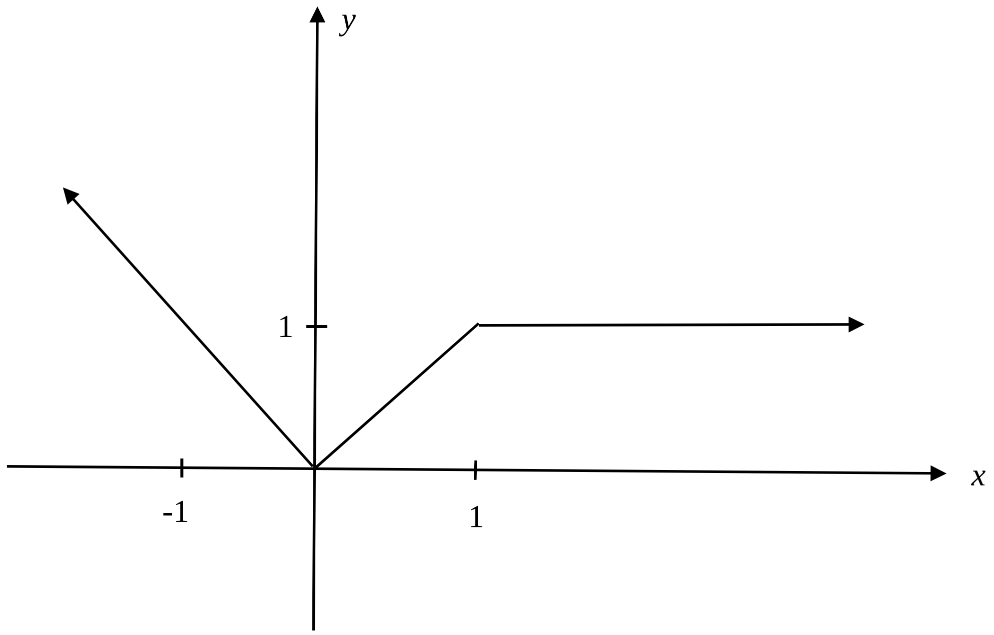
(ii) If this normal meets the hyperbola again at $Q(2q, \frac{2}{q})$ prove that $p^3q = -1$. **2**

(iii) Hence prove that there exists only one chord which is normal to the hyperbola at both ends and find its equation. **4**

Question 15 (15 marks) Start a NEW booklet. **Marks**

(a) Find the equation of the ellipse with centre the origin, which has a focus at $(2,0)$ and the corresponding directrix is $x = 4$. **3**

(b)



The diagram shows the graph of the function $y = f(x)$

Draw separate sketches of the following:

(i) $y = f(-x)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **1**

(iii) $y = f(|x|)$ **1**

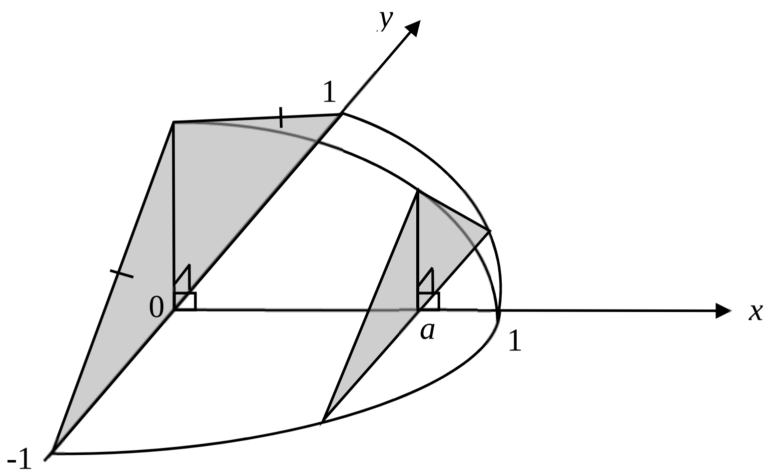
(iv) $y = \ln(f(x))$ **2**

(v) $y = e^{f(x)}$ **1**

(vi) $x = f(y)$ **1**

Question 15 continues on the next page.

(c) The base of a solid is the semi-circular region of radius 1 unit in the *x*-*y* plane as illustrated in the diagram below.



Each cross-section perpendicular to the *x*-axis is an isosceles triangle with each of the two equal sides three quarters the length of the third side..

(i) Show that the area of the triangular cross-section at $x = a$ is **2**

$$\frac{\sqrt{5}}{2}(1 - a^2).$$

(ii) Hence find the volume of the solid. **3**

Question 16 (15 marks) Start a NEW booklet. **Marks**

(a) $P(4,6)$ and $Q(14,24)$ are two points on the hyperbola **3**

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

M is the midpoint of PQ and $O(0,0)$ is the origin. The tangents to the hyperbola at P and Q intersect at the point R. Show that the points R, O and M are collinear.

You may assume that the tangent to this hyperbola at $T(x_1, y_1)$ has equation

$$\frac{x_1 x}{4} - \frac{y_1 y}{12} = 1$$

(Do NOT prove this.)

(b) A particle is moving so that $\ddot{x} = 18x^3 + 27x^2 + 9x$.

Initially $x = -2$ and the velocity, v, is -6. It is known that the velocity is always negative.

(i) Show that $v^2 = 9x^2(1 + x)^2$. **2**

(ii) Hence, or otherwise, show that **2**

$$\int \frac{1}{x(1+x)} dx = -3t$$

(iii) Find a, b such that **1**

$$\frac{1}{x(1+x)} \equiv \frac{a}{x} + \frac{b}{1+x}$$

(iv) Show that for some constant c, **2**

$$log_e \left(1 + \frac{1}{x}\right) = 3t + c$$

(v) Using this equation and the initial conditions, find x as a function of t. **2**

(c) The angles A, B and C are consecutive terms in an arithmetic series. Show that **3**

$\cos(A)\cos(C) - cos^2(B) = \sin(A)\sin(C) - sin^2(B)$.

END OF EXAM

Candidate Name/Number: ______________________________________

Multiple choice answer page. Fill in either A, B, C or D for questions 1-10.

<u>This page must be handed in with your answer booklets</u>

1.	
2.	
3.	
4.	
5.	

6.	
7.	
8.	
9.	
10.	

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

CARINGBAH HIGH EXTENSION II 2013

THSC SOLUTIONS

Q1 C	Q6 A
Q2 D	Q7 D
Q3 B	Q8 A
Q4 B	Q9 D
Q5 A	Q10 C

Q11

(a) (i) $\int \frac{t^2-2}{t^3}\,dt = \int \left(\frac{1}{t} - \frac{2}{t^3}\right)dt$

$= \int \left(\frac{1}{t} - 2t^{-3}\right)dt$

$= \ln|t| + \frac{1}{t^2} + c$

(ii) $\int x.e^x\,dx = xe^x - \int 1.e^x\,dx$ (u.v, u'.v)

$= xe^x - e^x + c$

(iii) $I = \int \frac{2x}{(x+1)(x+3)}\,dx$ Using partial fractions

$2x \equiv a(x+3) + b(x+1)$

$x=-1 \Rightarrow a=-1$

$x=-3 \Rightarrow b=3$

$\therefore I = \int \left(\frac{-1}{x+1} + \frac{3}{x+3}\right)dx$

$= 3\ln|x+3| - \ln|x+1| + c$

$= \ln\left|\frac{(x+3)^3}{x+1}\right| + c$

(b) $I = \int_4^{4.5} \frac{dx}{\sqrt{(x-3)(5-x)}}$

when $x=4.5,\ u=0.5$

$x=4,\ u=0$

If $u = x-4$

$du = dx$

$\int_{u=0}^{u=0.5} \frac{1.du}{\sqrt{(1+u)(1-u)}}$

$= \int_{u=0}^{u=\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\,du$

$= \left[\sin^{-1} u\right]_0^{1/2}$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$

(c) (i) $U_n = \left[-x^n.\cos x\right]_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1}.\cos x.dx$ (u.v, u'.v)

$= 0 + n\int_0^{\pi/2} x^{n-1}.\cos x.dx$

Now use parts a second time.

$= n\left[x^{n-1}.\sin x\right]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-2}.\sin x.dx$

$= n\left(\left(\frac{\pi}{2}\right)^{n-1}.\sin\frac{\pi}{2} - 0\right) - n(n-1)U_{n-2}$

$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)U_{n-2}$

(ii) $n=2 \Rightarrow U_2 = 2\left(\frac{\pi}{2}\right)^1 - 2.1.U_0$

$= \pi - 2\int_0^{\pi/2} \sin x.dx$

$= \pi + 2\left[\cos x\right]_0^{\pi/2}$

$= \pi + 2(0-1)$

$= \pi - 2$

Q12 (a) (i) $w = -1 + i\sqrt{3}$.

$w^2 = (-1+i\sqrt{3})(-1+i\sqrt{3})$

$= 1 - 3 - 2i\sqrt{3}$

$= -2 - 2i\sqrt{3}$

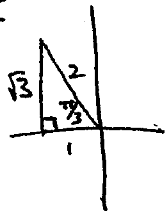
$\bar{w} = -1 - i\sqrt{3}$

$2\bar{w} = 2(-1-i\sqrt{3})$

$= -2 - 2i\sqrt{3}$

$= w^2$

(ii) $|w| = \sqrt{1+3}$ $= 2$ $\arg w = \frac{2\pi}{3}$



(iii) $\therefore w = 2 \operatorname{cis} \frac{2\pi}{3}$

$w^3 = 2^3 \operatorname{cis}\left(3 \cdot \frac{2\pi}{3}\right)$

$= 8 \operatorname{cis} 2\pi$

$= 8 \times 1$

$w^3 = 8$

$\therefore w^3 - 8 = 0$

(b) (i) Let $z = x + iy$.

$Re(z) = x$

$|z| = \sqrt{x^2+y^2}$

as $Re(z) = |z|$

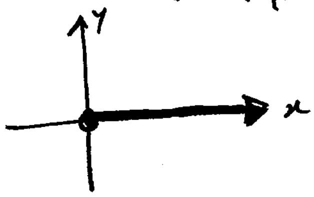
then $x = \sqrt{x^2+y^2}$

$x^2 = x^2 + y^2$

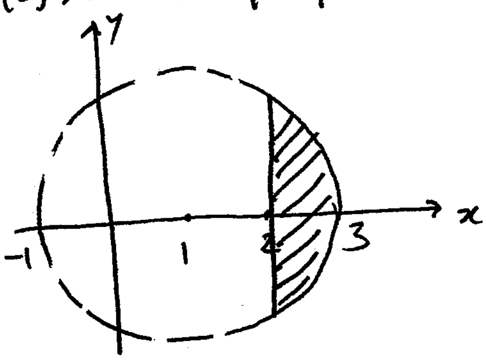
$y^2 = 0$

$\therefore y = 0$ ie x-axis

But $|z| \geq 0$ $\therefore$ only $x \not< 0$



(ii) $Re(z) \geq 2$ and $|z-1| \leq 2$



(c) $\therefore a(1+2i) + b(1+i) = (1+i)(1+2i)$

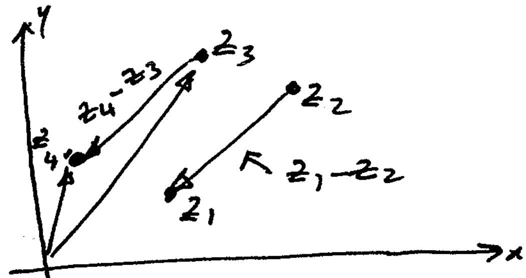
equating real parts $\Rightarrow$ $a + b = -1$

" imaginary parts $\Rightarrow$ $2a + b = 3$

$\therefore a = 4$

$b = -5$

(d) $\therefore z_1 - z_2 = z_4 - z_3$



If $z_1 - z_2 = z_4 - z_3$

then $|z_1 - z_2| = |z_4 - z_3|$

and $\arg(z_1 - z_2) = \arg(z_4 - z_3)$

$\therefore AB = CD$

and $AB \parallel CD$

$\therefore ABCD$ is a parallelogram.

Q13

(a) (i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (-b)^2 - 2 \times 1$

$= b^2 - 2$

(ii) real roots $\Rightarrow \alpha^2 + \beta^2 + \gamma^2 \geq 0$

but here $\alpha, \beta, \gamma \neq 0$

$\therefore \alpha^2 + \beta^2 + \gamma^2 > 0$

$\therefore b^2 - 2 > 0$

$\therefore b < -\sqrt{2}$ or $b > \sqrt{2}$

(iii) $2\alpha = x$ $\therefore \frac{x}{2} = \alpha$

in $\alpha^3 + b\alpha^2 + \alpha + 2 = 0$

$\therefore \left(\frac{x}{2}\right)^3 + b\left(\frac{x}{2}\right)^2 + \frac{x}{2} + 2 = 0$

$\frac{x^3}{8} + \frac{bx^2}{4} + \frac{x}{2} + 2 = 0$

$\therefore x^3 + 2bx^2 + 4x + 16 = 0$

(b) $P'(x) = 32x^3 - 108x^2 - 132x - 35$

$P''(x) = 96x^2 - 216x - 132$

$= 12(8x^2 - 18x - 11)$

$= 12(2x+1)(4x-11)$

$\therefore x = -1/2$ or $\frac{11}{4}$

$P(\frac{11}{4}) \neq 0$, $P(-1/2) = 0$

$\therefore P(x) = (2x+1)^3 Q(x)$

$= (2x+1)^3 \cdot (x-6)$

$\therefore x = -1/2, -1/2, -1/2, 6$

(c) (i) $\therefore z^5 - 1 = 0$

ie $(z-1)(z^4+z^3+z^2+z+1) = 0$

$z \neq 1$ so $z^4+z^3+z^2+z+1=0$

$z \neq 0$ so $\div$ by z^2

ie $z^2+z+1+\frac{1}{z}+\frac{1}{z^2}=0$

(ii) $\therefore \left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)+1=0$

$\left(z+\frac{1}{z}\right)^2 - 2 + \left(z+\frac{1}{z}\right)+1=0$

$\therefore x^2 + x - 1 = 0$

(iii) From $z^5 = 1$

$z_1 = cis\,\frac{2\pi}{5}$

$z_2 = cis\,\frac{4\pi}{5}$

$\frac{1}{z_2} = z_3 = cis\,\frac{6\pi}{5}$

$\frac{1}{z_1} = z_4 = cis\,\frac{8\pi}{5}$

These are the 4 roots of (i)

$z_5 = cis\,\frac{10\pi}{5} = 1$

from (ii) $x = x_1$ or x_2

$x_1 = z_1 + \frac{1}{z_1} = cis\,\frac{2\pi}{5} + cis\,-\frac{2\pi}{5}$

$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$+ \cos-\frac{2\pi}{5} + i\sin-\frac{2\pi}{5}$

as cos EVEN function and sin ODD function

$\therefore x_1 = 2\cos\left(\frac{2\pi}{5}\right)$

and $x_2 = z_2 + \frac{1}{z_2}$

$= 2\cos\left(\frac{4\pi}{5}\right)$

from (ii) $x_1 \cdot x_2 = -1$

$\therefore 2\cos\left(\frac{2\pi}{5}\right) \cdot 2\cos\left(\frac{4\pi}{5}\right) = -1$

$\therefore \cos\left(\frac{2\pi}{5}\right) \cdot \cos\left(\frac{4\pi}{5}\right) = -1/4$

Q14 (a) Let $\angle DAT = \beta$, Let $\angle ABC = \alpha$

(ii) $\angle DAT = \angle ACD$ ($\angle$s in alt. seg.)

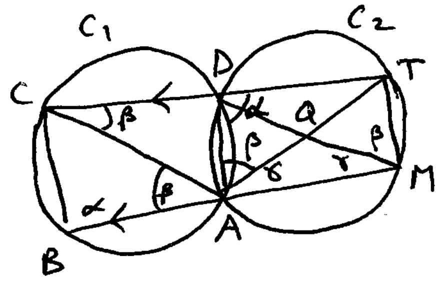
$\angle ACD = \angle BAC$ (alt $\angle$s, $CD \parallel AB$)

$\angle ABC = \angle ADT$ (ext $\angle$ cyclic quad ABCD)

(iii) $\angle MAT = \angle MDT$ ($\angle$s in same seg)

$\angle MDT = \angle AMD$ (alt $\angle$s, $DT \parallel AM$)

Let $\angle MAT = \gamma$



$\therefore \triangle QMA$ is isos.

(iv) $\angle DMT = \angle DAT$ ($\angle$s in same seg)

$\therefore \angle AMT = \beta + \gamma$

$\angle ATD = \angle AMD$ ($\angle$s in same seg)

$= \gamma$

$\therefore$ From $\triangle ADT$ $\alpha + \beta + \gamma = 180°$

$\therefore \angle BCA = \gamma$

$\therefore \angle BCD = \beta + \gamma = \angle AMT$

$\therefore$ BCTM is a parallelogram (1 pair of sides parallel plus one pair of opp. $\angle$s equal).

$\therefore CB = TM$.

(b) (i)

$$y = \frac{4}{x}$$

$$y' = -\frac{4}{x^2}$$

$$\therefore m_{T_p} = -\frac{4}{4p^2} = -\frac{1}{p^2}$$

$$m_{N_p} = p^2$$

$$\therefore y - \frac{2}{p} = p^2(x-2p)$$

$$py - 2 = p^3x - 2p^4$$

$$p^3x - py = 2(p^4-1) \quad \text{— (*)}$$

(ii) $\therefore$ Q satisfies $xy=4$ then $y = \frac{4}{x}$

so $p^3 \cdot x - p \cdot \frac{4}{x} = 2(p^4-1)$

or $p^3x^2 - 2(p^4-1)x - 4p = 0$

$$\therefore x = \frac{2(p^4-1) \pm \sqrt{4(p^4-1)^2 + 16p^4}}{2p^3}$$

$$= \frac{p^4-1 \pm \sqrt{(p^4-1)^2 + 4p^4}}{p^3}$$

$$= \frac{p^4-1 \pm (p^4+1)}{p^3}$$

$$= \frac{p^4-1+p^4+1}{p^3} \text{ or } \frac{p^4-1-p^4-1}{p^3}$$

$$= \frac{2p^4}{p^3}, \quad \frac{-2}{p^3}$$

$$= 2p$$

Now if $x = 2p$ then we have $P\left(2p, \frac{2}{p}\right)$

so $Q\left(2q, \frac{2}{q}\right) \Rightarrow 2q = \frac{-2}{p^3}$

or $p^3q = -1$

(iii) By symmetry $q^3p = -1$ also

$$\therefore p^3q = pq^3$$

or $p^3q - q^3p = 0$

$$pq(p^2-q^2) = 0$$

$$pq(p-q)(p+q) = 0$$

$\therefore pq = 0$ — not possible

or $p = q$ — not possible

$\therefore p = -q$.

$$\therefore q^3 \cdot -q = -1$$

$$q^4 = 1$$

$$\therefore q = \pm 1$$

$$\therefore P \equiv (\pm 2, \pm 2)$$

$\therefore$ chord is $y = x \; : \; -2 \le x \le 2$

Q15

(a)

$ae = 2$ — (1)

$a/e = 4$ — (2)

(1) × (2) $\Rightarrow a^2 = 8$

(1) ÷ (2) $\Rightarrow e^2 = \frac{1}{2}$

ellipse $\Rightarrow e^2 = 1 - \frac{b^2}{a^2}$

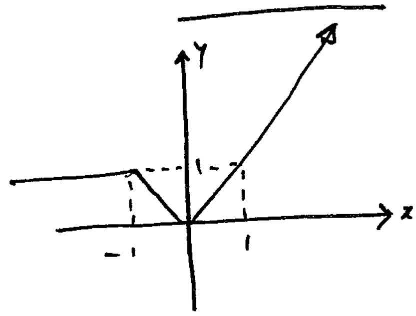
$$\frac{1}{2} = 1 - \frac{b^2}{8}$$

$$4 = 8 - b^2$$

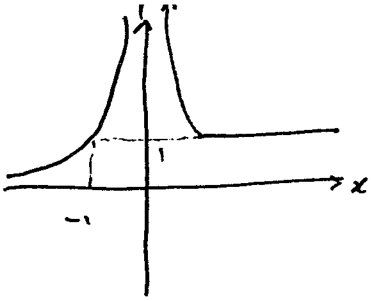
$$b^2 = 4$$

$$\therefore \frac{x^2}{8} + \frac{y^2}{4} = 1$$

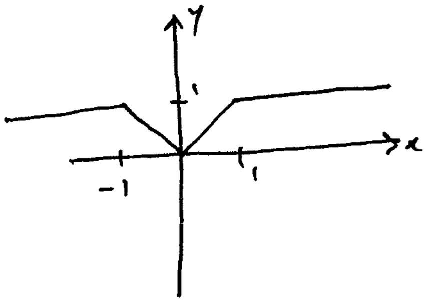
(b) (i)



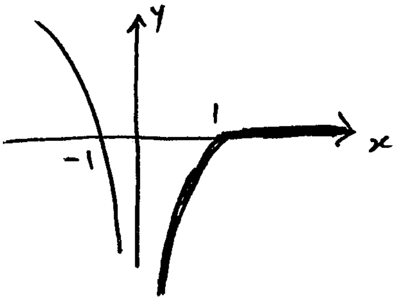
(ii)



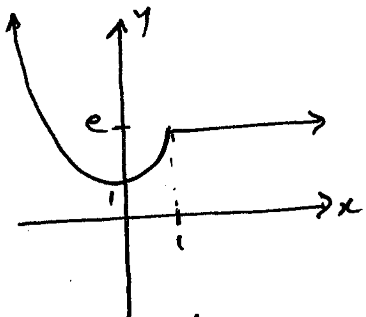
(iii)



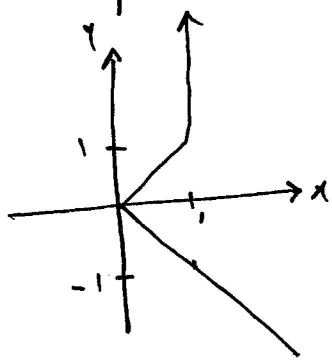
(iv)



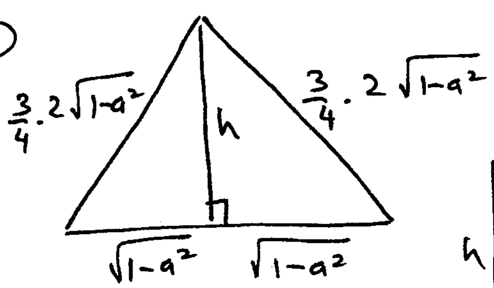
(v)

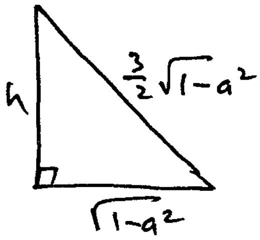


(vi)



(c)





$$h^2 = \frac{9}{4}(1-a^2) - (1-a^2)$$
$$= \frac{5}{4}(1-a^2)$$

$$\therefore h = \frac{\sqrt{5}}{2}\sqrt{1-a^2}$$

$$\therefore \text{Area} = \frac{1}{2}\left(2\sqrt{1-a^2}\right) \cdot \frac{\sqrt{5}}{2}\sqrt{1-a^2}$$

$$= \frac{\sqrt{5}}{2}(1-a^2)$$

(ii)
$$V = \frac{\sqrt{5}}{2}\int_{x=0}^{x=1}(1-x^2)\,dx$$
$$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$$
$$= \frac{\sqrt{5}}{2}\left(1 - \frac{1}{3} - (0-0)\right)$$
$$= \frac{\sqrt{5}}{3}\ u^2$$

Q16 (a) $O \equiv (0,0)$ $M \equiv \left(\frac{18}{2}, \frac{30}{2}\right) \equiv (9,15)$

$$\text{Tangent}_P \equiv T_P \equiv \frac{4x}{4} - \frac{6y}{12} = 1$$
$$\therefore x - \frac{y}{2} = 1 \quad \text{— ①}$$

$$T_Q \equiv \frac{14x}{4} - \frac{24y}{12} = 1$$
$$\therefore \frac{7x}{2} - 2y = 1 \quad \text{— ②}$$

② − 4×①
$$\frac{7x}{2} - 4x = 1 - 4$$
$$7x - 8x = -6$$
$$-x = -6$$
$$x = 6$$

sub into ②
$$6 - \frac{y}{2} = 1$$
$$y/2 = 5$$
$$y = 10$$

$$\therefore R \equiv (6,10)$$

$$m_{RO} = \frac{10}{6} = \frac{5}{3}$$
$$m_{MO} = \frac{15}{9} = \frac{5}{3}$$
$\}$ $m_{RO} = m_{MO}$ $\Rightarrow$ RMO collinear

(i) $\int \frac{d}{dx} \frac{1}{2}v^2 \, dx = \int (18x^3 + 27x^2 + 9x) \, dx$

$\frac{1}{2}v^2 = \frac{9x^4}{2} + 9x^3 + \frac{9x^2}{2} + c$

$v^2 = 9x^4 + 18x^3 + 9x^2 + c'$

$x = -2, v = -6 \quad \therefore c' = 0$

$\therefore v^2 = 9x^2(x^2 + 2x + 1)$

$= 9x^2(x+1)^2$

(ii) $\therefore v = 3x(1+x)$ or $-3x(1+x)$

here $x = -2 \Rightarrow v = -6 \times -1 = 6$ not -6	here $x = -2$, $v = 6 \times -1 = -6$

$\therefore v = -3x(1+x)$

$\therefore \frac{dx}{dt} = -3x(1+x)$

$\frac{dt}{dx} = -\frac{1}{3} \frac{1}{x(1+x)}$

integrate both sides w.r.t x gives

$t = -\frac{1}{3} \int \frac{1}{x(1+x)} dx$

or $-3t = \int \frac{1}{x(1+x)} dx$

(iii) $1 \equiv a(1+x) + bx$

$x = 0 \Longrightarrow a = 1$

$x = -1 \Longrightarrow b = -1$

(iv) $\therefore -3t = \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx$

$= \ln|x| - \ln|1+x| + c$

$3t = \ln\left|\frac{1+x}{x}\right| + c$

$3t + c' = \ln\left|1 + \frac{1}{x}\right|$

(v) $t = 0 \quad x = -2$

$\therefore \ln\left(\frac{1}{2}\right) = c'$

$\therefore 1 + \frac{1}{x} = e^{3t + \ln 1/2}$

$= \frac{1}{2}e^{3t}$

$\frac{1}{x} = \frac{1}{2}e^{3t} - 1$

$x = \frac{1}{\frac{1}{2}e^{3t} - 1}$

$x = \frac{2}{e^{3t} - 2}$

(c) AP $\Rightarrow A, B, C \equiv B-d, B, B+d$

so we need only show.

$\cos A \cos C - \sin A \sin C = \cos^2 B - \sin^2 B$

$= \cos(2B)$

$LHS = \cos(A+C)$

$= \cos(B - d + B + d)$

$= \cos(2B)$

$= RHS$