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# 2003
## TRIAL HIGHER SCHOOL CERTIFICATE

# MATHEMATICS
## Extension 2

CARINGBAH HIGH
OMNIA DILIGENTIA VINCIT

**General Instructions**

Reading Time: 5 minutes
Working Time: 3 hours

- Attempt all questions
- Start each question on a new page
- Each question is of equal value
- Show all necessary working.
- Marks may be deducted for careless work or incomplete solutions
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- This examination paper must not be removed from the examination room

**Question 1. ( 15 marks ) Start a new page.** **Marks**

a) Find $\int \sec^2 x\,(\tan^2 x + 2)\,dx$. 2

b) Find $\int \frac{5}{x^2 + 6x + 13}\,dx$. 2

c) Use $t = \tan\left(\frac{x}{2}\right)$ to find $\int \frac{dx}{1 + \sin x + \cos x}$. 3

d) Find $\int \frac{e^{2x}}{\left(e^x + 1\right)^2}\,dx$ using the substitution $u = e^x + 1$ 2

e) Find $\int 3^x\,dx$. 1

f) i) Let $I_n = \int_0^1 x^n e^x\,dx$ where $n \geq 0$. Show that 3

$I_n = e - nI_{n-1}$ for $n \geq 1$.

ii) Hence evaluate $\int_0^1 x^3 e^x\,dx$. 2

**Question 2. ( 15 marks ) Start a new page** **Marks**

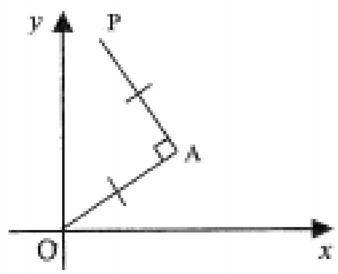
a) Let $z = 3 - 4i$ and $w = 2 + 5i$. Express the following in the form $x + iy$, where $x$ and $y$ are real numbers.

i) $z^2$ 1

ii) $i^3 \frac{z}{w}$ 2

b) Find all the complex numbers $z = a + ib$, where $a$ and $b$ are real, such that $\left|z^2\right| + i\,\overline{z} = 11 + 3i$ 3

c)




The point A in the complex plane corresponds to the complex number $z$. The triangle $OAP$ is a right angled isosceles triangle.

i) Find in terms of $z$ the complex number corresponding to the point $P$. 1

ii) Let $M$ be the midpoint of $OP$. What complex number corresponds to $M$ ? 1

d) i) Express $3 - 3i$ in modulus-argument form. 1

ii) Hence evaluate $(3 - 3i)^7$, expressing it in the form $a + ib$ where $a$ and $b$ are real numbers. 2

e) i) On the same diagram, draw a neat sketch of the locus specified by: 2

$\alpha$) $\left|z - (5 + 4i)\right| = 4$

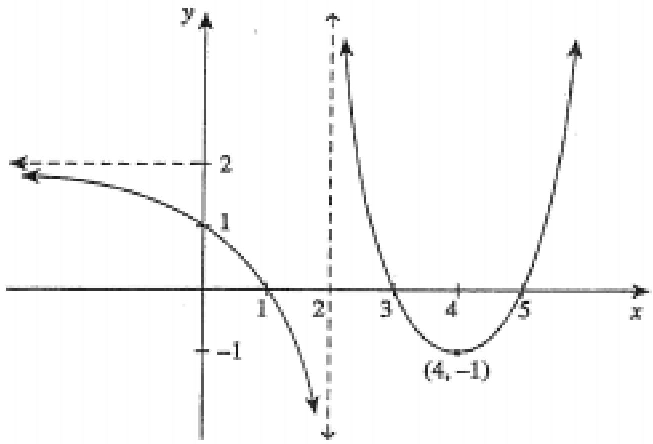
$\beta$) $\left|z + 4\right| = \left|z - 6\right|$

ii) Hence write down the value of $z$ which simultaneously satisfies $\left|z - (5 + 4i)\right| = 4$ and $\left|z + 4\right| = \left|z - 6\right|$ 1

iii) Use your diagram in (i) to determine the value(s) of $k$ for which the simultaneous equations $\left|z - (5 + 4i)\right| = 4$ and $\left|z - 4i\right| = k$ have exactly one solution for $z$. 1

**Question 3.** **( 15 marks ) Start a new page.** **Marks**

a) The graph of $y = f(x)$ is drawn below.




As $x \to -\infty, f(x) \to 2$. The line $x = 2$ is a vertical asymptote. The $y$-intercept is $y = 1$ and the $x$-intercepts are $x = 1, x = 3$ and $x = 5$.
Draw **separate** ***half-page*** sketches of the graphs of the following:

i) $y = |f(x)|$ 2

ii) $y = f(|x|)$ 2

iii) $y = \dfrac{1}{f(x)}$ 2

iv) $y = \tan^{-1}[f(x)]$ 2

b) i) Find the coordinates and the nature of the stationary points on the curve $y = x^3 + 6x^2 + 9x + k$ where $k$ is real. 2

ii) Hence find the set of values of $k$ for which the equation $x^3 + 6x^2 + 9x + k = 0$ has three real and different roots. 2

c) i) Find the domain and range of the function $f(x) = \tan^{-1}(e^x)$. 1

ii) Sketch the curve $f(x) = \tan^{-1}(e^x)$ showing any intercepts on the coordinate axes and the equations of any asymptotes. 2

**Question 4. ( 15 marks ) Start a new page.** **Marks**

a) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$, has eccentricity $e=\frac{1}{2}$.
The point $P(2,3)$ lies on the ellipse.

i) Find the values of $a$ and $b$. 3

ii) Sketch the graph of the ellipse showing clearly the intercepts on the axes and the coordinates of the foci. 2

b) The normal at the point $P\left(cp,\frac{c}{p}\right)$ on the hyperbola $xy=c^2$ meets the $x$-axis at $Q$. Also let $M$ be the midpoint of $PQ$.

i) Show that the normal at $P$ has the equation $p^3x-py=c\left(p^4-1\right)$ 2

ii) Show that $M$ has coordinates $\left(\frac{c\left(2p^4-1\right)}{2p^3},\frac{c}{2p}\right)$ 2

iii) Hence or otherwise, find the equation of the locus of $M$. 3

c) The polynomial $P(z)$ is defined by $P(z)=z^4-2z^3-z^2+2z+10$.

i) Given that $z=2-i$ is a root of $P(z)$ write down another root giving a reason for your answer. 1

ii) Hence, express $P(z)$ as a product of real quadratic factors. 2

**Question 5. ( 15 marks ) Start a new page.** **Marks**

a) i) Suppose that the polynomial $P(x)$ has a double zero at $x = \alpha$. Prove that $P'(x)$ also has a zero at $x = \alpha$. 2

ii) The polynomial $P(x) = x^4 + ax^3 + bx + 21$ has a double zero at $x = 1$. Find the values of $a$ and $b$. 2

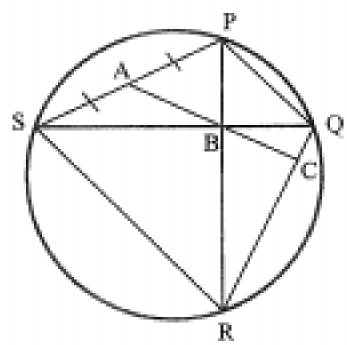
b) i) The equation $x^3 + px^2 + qx + r = 0$ ( where $p, q, r$ are non zero ) has roots $\alpha, \beta, \gamma$ such that $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are consecutive terms in an arithmetic sequence. 3

Show that $\beta = \frac{-3r}{q}$.

ii) The equation $x^3 - 26x^2 + 216x - 576 = 0$ has roots $\alpha, \beta, \gamma$ such that $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are consecutive terms in an arithmetic sequence. 3

Find the values of $\alpha, \beta, \gamma$.

c)




PQRS is a cyclic quadrilateral. The diagonals PR and SQ intersect at right angles at B. A is the midpoint of PS. AB produced meets QR at C.

Let $\angle ABP = \alpha$. Using the larger diagram provided to indicate angles, show that

i) B, P and S are concyclic points. 1

ii) $\angle APB = \angle ABP$. 1

iii) AC is perpendicular to QR. 3

**Question 6. ( 15 marks ) Start a new page.** **Marks**

a) If $\bar{z}_1 + \bar{z}_2 = 5 + 2i$, find $z_1 + z_2$ 1

b) The arc of the curve $y = x(2 - x^2)$ from $x = 0$ to $x = 1$ is rotated about $y$ axis. Find by using **cylindrical shells** the volume of the solid formed. 4

c) i) Show that $a^2 + b^2 > 2ab$, where $a$ and $b$ are distinct positive real numbers. 1

ii) Hence show that $a^2 + b^2 + c^2 > ab + bc + ca$, where $a$, $b$ and $c$ are distinct positive real numbers. 2

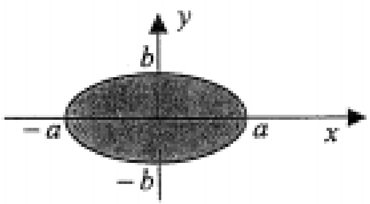
iii) Hence or otherwise, prove that $\dfrac{a^2b^2 + b^2c^2 + c^2a^2}{a + b + c} > abc.$ 2

d) i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$ using the substitution $u = a - x$ 2

ii) Hence evaluate $\int_0^2 x^2\sqrt{2 - x}\,dx$, writing your answer in the form $a\sqrt{b}$. 3

**Question 7. ( 15 marks ) Start a new page.** **Marks**

a)

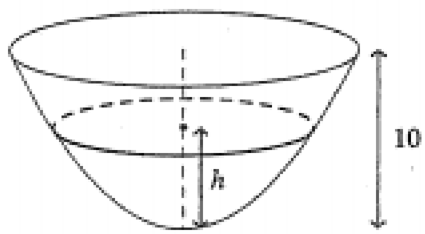



The diagram shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with major diameter $2a$ and minor diameter $2b$.

i) Show that the shaded area of the ellipse is given by $\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$. 2

ii) Hence show that the shaded area is $\pi ab$ square units. 2

iii)




The diagram above shows a solid of height 10 cm. At height $h$ cm above the vertex, the cross-section of the solid is an ellipse with major diameter $10\sqrt{h}$ cm and minor diameter $8\sqrt{h}$ cm.

$\alpha$) Show that the cross-section at height $h$ cm above the vertex has area $20\pi h \text{ cm}^2$. 2

$\beta$) Find the volume of the solid in exact form. 2

b) If $\alpha, \beta, \gamma$ are the roots of the equation $2x^3-7x^2+5x-3=0$,

i) Show that the equation with roots $\alpha^2, \beta^2, \gamma^2$ is given by $4x^3-29x^2-17x-9=0$ 2

ii) Hence evaluate $\alpha^3+\beta^3+\gamma^3$. 1

c) i) Expand $(\cos\theta + i\sin\theta)^3$ into powers of $\cos\theta$ and $\sin\theta$. 1

ii) By using De Moivres Theorem show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. 2

iii) Hence find the exact value of $4\cos^3\left(\frac{\pi}{12}\right)-3\cos\left(\frac{\pi}{12}\right)$. 1

**Question 8. ( 15 marks ) Start a new page.** **Marks**

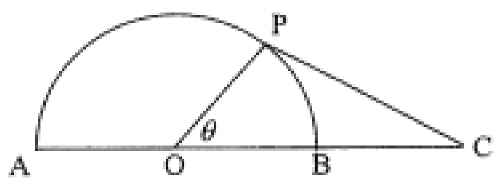
a) Find ***all*** solutions in radians of the equation 3

$$\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \frac{3}{4}$$

b) **For this question assume that tidal motion is simple harmonic.**

On a certain day, the depth of water in a harbour at high tide at 5 am is 9 metres. At the following low tide at 11:20 am the depth is 3 metres. Find the latest time before noon that a ship can enter the harbour if a minimum depth of 7.5 metres is required. ( **Show all reasoning** ). 4

c)




In the diagram above the fixed points A, O, B and C are on a straight line such that AO = OB = BC = 1unit. The points A and B are also joined by a semicircle and P is a variable point on this semicircle such that $\angle$ POC $= \theta$.

$R$ is the region bounded by the arc AP of the semicircle and the straight lines AC and PC.

i) Show that the area $S$ of $R$ is given by: $S = \frac{\pi}{2} - \frac{\theta}{2} + \sin\theta$. 1

ii) Find the value of $\theta$ for which $S$ is a maximum. 2

iii) Show that the perimeter $L$ of $R$ is given by: 2

$L = 3 + \pi - \theta + \sqrt{5 - 4\cos\theta}$.

iv) Show that $L$ has just one stationary point and that it occurs at the same value of $\theta$ for which $S$ is a maximum. 2

v) Hence find the greatest value of $L$ in the interval $0 \le \theta \le \pi$. 1

***END OF PAPER***

# STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$