CSSA - NSW

4 Unit Mathematics

Trial HSC Examination 1986

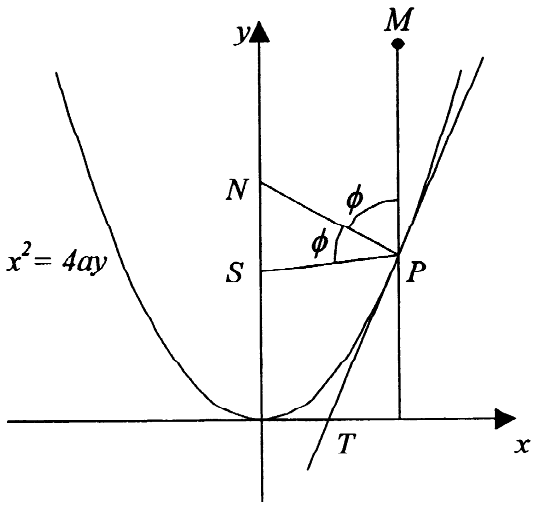
1. **(i)** Use partial fractions to find the integral $\int \frac{x^2}{(x+1)(x+2)}\,dx$.
(ii) Use the substitution $x = \frac{\pi}{2} - u$ to show that $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$ and hence determine their value.
(iii) Find the integral $\int \frac{e^x + e^{2x}}{1 + e^{2x}}\,dx$.
(iv) Find $\int_0^{\frac{\pi}{4}} x \sec^2 x\,dx$.

2. **(i)** If $P(x) = x^3 - 6x^2 + 9x + \alpha$, where α is a constant, find the values of x for which $P'(x) = 0$. Determine the values of α such that the equation $P(x) = 0$ has a repeated root. By sketching the graphs of $y = P(x)$ for these values of α find the set of values of α for which the equation $P(x) = 0$ has only one root.
(ii) Sketch the graph of the function $y = \frac{x^2 - x + 1}{(x-1)^2}$ showing clearly the coordinates of any points of intersection with the x-axis and the y-axis, the coordinates of any turning points and the equations of any asymptotes. (There is no need to investigate points of inflexion).

3. **(i)** The complex numbers $z_1 = \frac{a}{1+i}$ and $z_2 = \frac{b}{1+2i}$ where a and b are real, are such that $z_1 + z_2 = 1$. Find the values of a and b.
(ii) The complex number z has modulus r and argument θ where $0 < \theta < 2\pi$. Find in terms of r and θ the modulus and argument of:
(a) z^2;
(b) $\frac{1}{z}$;
(c) iz.
(iii) If $z_1 = 3 + 4i$ and $|z_2| = 13$ find the greatest value of $|z_1 + z_2|$. If $|z_1 + z_2|$ has its greatest value and also $0 < \arg z_2 < \frac{\pi}{2}$ express z_2 in the form $a + ib$ where a and b are real.
(iv) If $z = x + iy$, where x and y are real, find and sketch the locus of the set of points $P(x, y)$ such that $\Re(z - \frac{1}{z}) = 0$.

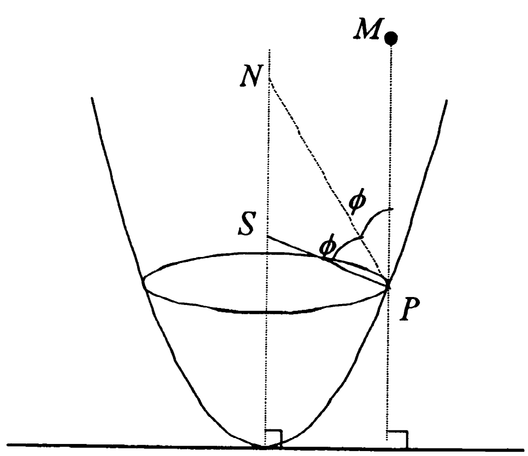
4. **(i)** Show that the curves $x^2 - y^2 = c^2$ and $xy = c^2$ cross at right angles.
(ii) Show that the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $bx\sec\theta - ay\tan\theta = ab$, and deduce that the normal there has equation $by\sec\theta + ax\tan\theta = (a^2 + b^2)\sec\theta\tan\theta$. The tangent and the normal cut the y axis at A and B respectively. Show that the circle on AB as diameter passes through the foci of the hyperbola. (It is enough to show that the circle passes through one focus and then to use symmetry).

5. (i)



$P(2ap, ap^2)$, where $0 < p < 1$, is a point on the parabola $x^2 = 4ay$ with focus S. The normal to the parabola at P meets the y-axis at N. MP is parallel to the axis of the parabola. The tangent to the parabola at P meets the x-axis at T. $\angle SPN = \phi$. Assuming the standard property of the parabola that NP bisects $\angle SPM$, show geometrically that $\tan\phi = p$.

(ii) A smooth bowl has the same shape as a paraboloid shell formed by rotating around the y-axis the curve $x = 2p, y = p^2$, where $0 \leq p \leq 1$. P, S, N and M are as defined in part **(i)** above.



A particle of mass m is attached by a light inextensible string of length 1.25 units to a fixed point at the focus S of the bowl. The particle is observed to be travelling in a horizontal circle with constant angular velocity ω while staying in contact with the bowl, the string remaining taut throughout the motion.

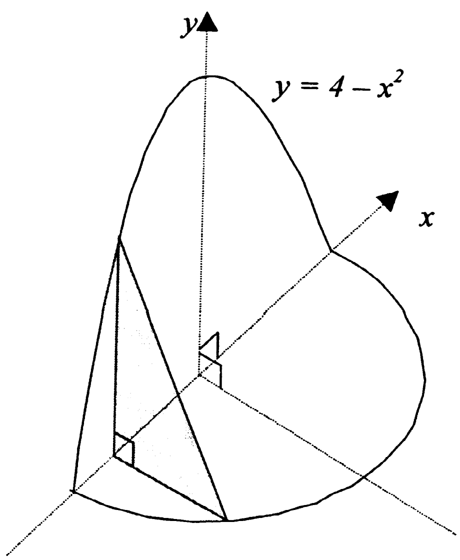
(a) Find SP in terms of p and hence show that the path of the particle passes through the point P with the parameter $\frac{1}{2}$.

(b) Copy the diagram and show all the forces on the particle when at P.

(c) Show that the tension in the string is $m(2\omega^2 - g)$.

(d) What happens when ω takes the critical value $\sqrt{\frac{1}{2}g}$? Could the particle be observed to travel around the bowl in a horizontal circle if $\omega = \omega_1$, where $\omega_1 < \sqrt{\frac{1}{2}g}$? (Give reasons for your answer).
(e) Is there any upper limit to values of ω for which the described motion could be observed? Discuss briefly.

6. **(i)** If $z = \cos\theta + i\sin\theta$ show that $z^n = \frac{1}{z^n} = 2\cos n\theta$ and hence show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.
(ii)



The solid shown sits on a semi-circular base of radius 2 units. Vertical cross sections perpendicular to the diameter of the semi-circle are right angled triangles, the height of the the triangles being bounded by the parabola $y = 4 - x^2$ as shown. By slicing at right angles to the x-axis as indicated in the diagram, show that the volume of the solid isgiven by $V = \int_0^2 (4 - x^2)^{\frac{3}{2}}\, dx$. Hence calculate the volume of the solid.

7. **(i)** ADB is a straight line with $AD = a$ and $DB = b$. A circle is drawn on AB as diameter. DC is drawn perpendicular to AB to meet the circle at C. By using similar triangles show that $DC = \sqrt{ab}$. Deduce geometrically that if a and b are positive real numbers then $\sqrt{ab} \leq \frac{a+b}{2}$.
(ii) A cylinder of height h and radius r is inscribed in a sphere of radius R. Show that $r = \sqrt{R^2 - \frac{h^2}{4}}$ and hence find the greatest volume of the cylinder in terms of R.
(iii) If ABC is a triangle show that $\frac{\sin A + \sin B}{\cos A + \cos B} = \cot\frac{C}{2}$.

8. **(i)** ABC is a triangle. D is the point on AB which divides AB internally in the ratio $m : n$. In addition $\angle ACD = \alpha, \angle BCD = \beta$, and $\angle CDB = \theta$. Express $\angle CAD$ and $\angle CBD$ in terms of α, β and θ. By using the sine rule in each of triangle CAD and triangle CBD show that $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$.

(ii) A polynomial $P(x)$ is divided by $x^2 - a^2$ where $a \neq 0$ and the remainder is $px + q$. Show that $p = \frac{1}{2a}\{P(a) - P(-a)\}$ and $q = \frac{1}{2}\{P(a) + P(-a)\}$. Find the remainder when the polynomial $P(x) = x^n - a^n$ is divided by $x^2 - a^2$ for the cases
(a) n even
(b) n odd.
(iii) If the sides of a triangle are in arithmetic progression with first term 1 and common difference d find the set of possible values of d.