CSSA of NSW

1991 HSC Trial 4 Unit Mathematics

1. (a) (i) If $g(x) = \frac{1}{f(x)}$ find $g'(x)$ in terms of $f(x)$ and $f'(x)$, and deduce that the x-coordinates of the stationary points of $y = g(x)$ are the x-coordinates of the stationary points of $y = f(x)$ for which $f(x)$ is non-zero.

(ii) Compare the nature of the stationary point of $y = f(x)$ and $y = g(x)$ which have the same x coordinate.

(iii) Describe the behaviour of $y = g(x)$ near the zeros of $y = f(x)$, and as $f(x)$ approaches $\pm\infty$.

(b) (i) Sketch $y = x \ln x$, showing any turning points.

(ii) Deduce that $x \ln x = 1$ has one root, and this root lies between $\sqrt{e}$ and e.

(iii) Show that if Newton's method is used to solve $x \ln x = 1$, with the first approximation to the root being a_1, then the next approximation in the sequence is $a_2 = \frac{1+a_1}{1+\ln a_1}$.

(iv) Hence approximate the root of $x \ln x = 1$ to two decimal places, using an integer $a_1, \sqrt{e} < a_1 < e$, as the first approximation.

(v) On the same diagram as the sketch in **(i)**, sketch $y = \frac{1}{x \ln x}$, showing any turning points and asymptotes, and the approximate intersection point of the two curves.

2. (a) The complex number z and its conjugate $\overline{z}$ satisfy the equation $z\overline{z} + 2iz = 12 + 6i$. Find the possible values of z.

(b) On an Argand diagram shade the region containing all points representing comlex numbers z such that $\Re(z) \leq 1$ and $0 \leq \arg(z + i) \leq \frac{\pi}{4}$.

(c) (i) Express $z_1 = \frac{7+4i}{3-2i}$ in the form $a + ib$ where a and b are real.

(ii) On an Argand diagram sketch the locus of the point representing the complex number z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of $|z|$ subject to this condition.

(d) The complex number $z = x + iy$, x and y real, is such that $|z - 1| = \Im(z)$.

(i) Show that the locus of the point P representing z has Cartesian equation $y = \frac{1}{2}(x^2 + 1)$. Sketch this locus.

(ii) Find the gradients of the tangents to this curve which pass through the origin, hence find the set of possible values of the principal argument of z (i.e., $-\pi < \arg z \leq \pi$).

3. (a) Find: **(i)** $\int \frac{1}{3+2x-x^2}\, dx$; **(ii)** $\int \frac{1}{e^x+e^{-x}}\, dx$.

(b) (i) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\, dx$ using the substitution $t = \tan \frac{x}{2}$.

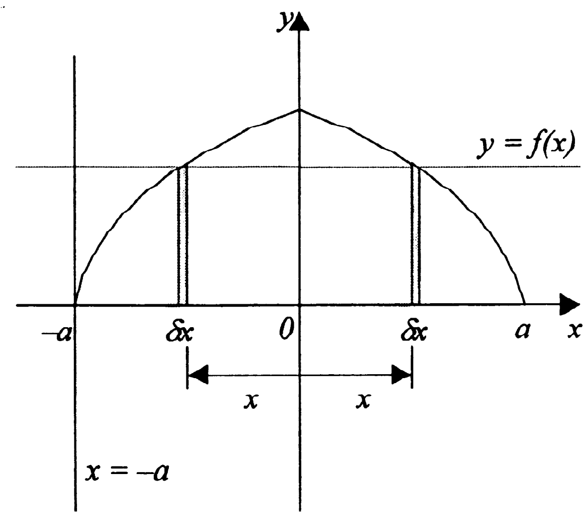
(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\, dx$ using the substitution $u = \frac{\pi}{2} - x$.

(c) (i) Let $I_n = \int_1^e x(\ln x)^n\, dx,\ n = 0, 1, 2, 3, \ldots$. Use integration by parts to show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}, n = 1, 2, 3, \ldots$.

(ii) The area bounded by the curve $y = \sqrt{x}(\ln x)^2,\ x \leq 1$, the x-axis and the line $x = e$ is rotated through 2π radians about the x-axis. Find the exact value of the volume of the solid of revolution so formed.

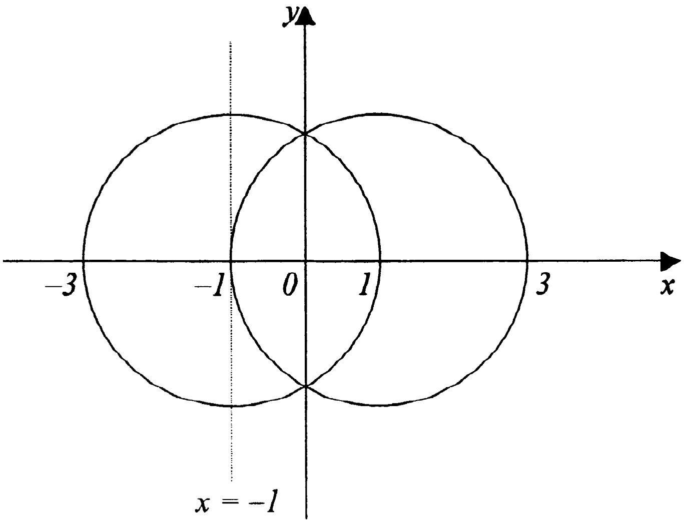
4. (a) $P(a\cos\theta, b\sin\theta), Q(a\sec\theta, b\tan\theta)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ respectively. M and N are the feet of the perpendiculars from P, Q respectively to the x-axis. $0 < \theta < \frac{\pi}{2}$, and QP produced meets the x-axis in K. A is the point with coordinates $(a, 0)$.
(i) Using without proof the similarity of $\triangle KPM$ and $\triangle KQN$, show that $\frac{KM}{KN} = \cos\theta$, and hence show that K has coordinates $(-a, 0)$.
(ii) Sketch the ellipse and hyperbola showing the positions of P, Q, M, N, A and K.
(iii) Show that the tangent to the ellipse at P has equation $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ and deduce that the tangent passes through N.
(iv) Given that the tangent to the hyperbola at Q has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$, show that this tangent passes through M.
(v) Show that the tangents PN and QM and the common tangent at A are concurrent, and show that the point of concurrence is $T(a, b\tan\frac{\theta}{2})$.
(vi) If the common tangent at A meets QP in V, show that T is the midpoint of AV.
(b) The result in **(a) (i)** provides a method of constructing the hyperbola $\frac{x^2}{4} - y^2 = 1$ from the auxiliary circle $x^2 + y^2 = 4$, and the ellipse $\frac{x^2}{4} + y^2 = 1$. Indicate why this is so on a new sketch by using the auxiliary circle to construct one such pair of points P, Q each with parameter $\theta,\ 0 < \theta < \frac{\pi}{2}$.

5. (a)



$f(x)$ is an even function. The area bounded by the curve $y = f(x)$ and the x-axis is rotated around the line $x = -a$. The strips of width δx shown in the diagram will form cylindrical shells of the same height.
(i) Find the sum of the approximating volumes of these two cylindrical shells for δx small.
(ii) Show that the volume of the solid is given by $V = 4\pi a \int_0^a f(x)\ dx$.

(iii) The centres of two circles, each of radius 2 cm, are 2 cm apart. The region common to the two circles is rotated around one of the tangents to this region which is perpendicular to the line joining the centres.



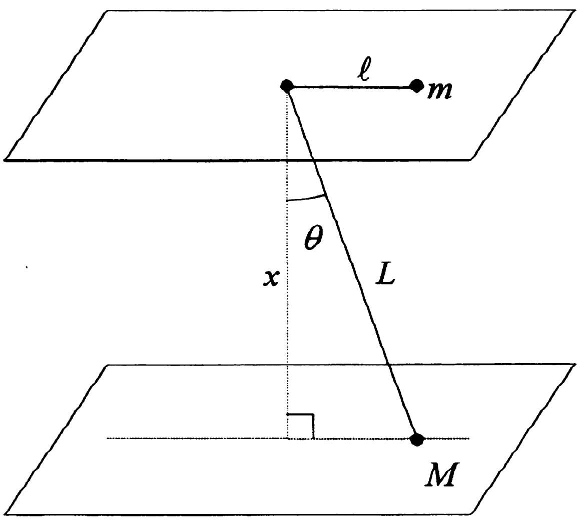
Show that the volume of the solid formed is given by $V = 8\pi \int_0^1 \sqrt{4-(x+1)^2}\,dx$ and hence find the exact volume of the solid.

(b) (i) If $a > 0$, show that $a + \frac{1}{a} \geq 2$, with equality if and only if $a = 1$.

(ii) Deduce that for $a > 0, b > 0, (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$, and state the condition for equality.

(iii) If $a > 0, b > 0, c > 0$, deduce that $(a+b)(b+c)(c+a) \geq 8abc$, and state the condition for equality.

6.

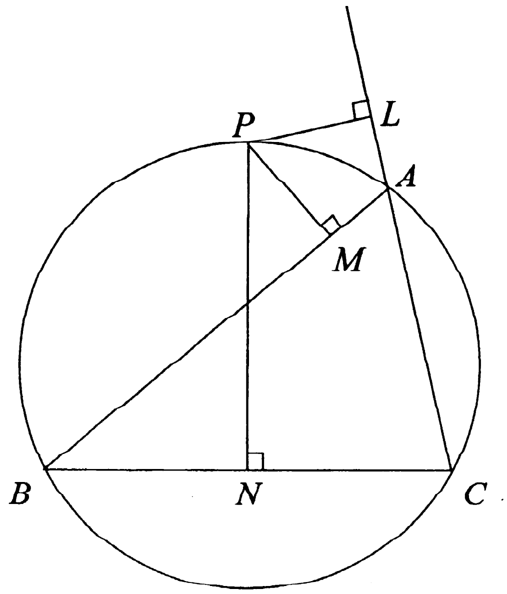


Two particles are connected by a light inextensible string passes through a small hole with smooth edges in a smooth horizontal table. One particle of mass m travels in a circle on the table with constant angular velocity ω. The second particle of mass M travels in a circle with constant angular velocity Ω on a smooth horizontal

floor distance L makes an angle θ with the vertical.
(a) (i) Draw diagrams showing the forces on each particle.
(ii) If the floor exerts a force N on the lower particle, show $N = M(g - x\Omega^2)$. State the maximum possible value of Ω for the motion to continue as described. What happens if Ω exceeds this value?
(iii) By considering the tension force in the string, show $\frac{L}{l} = \frac{m}{M}\left(\frac{\omega}{\Omega}\right)^2$.
(iv) If the lower particle exerts zero force on the floor, show that the tension T in the string is given by $T = \frac{MgL}{x}$.
(b) The table is 80 cm high, and the string is 1·5 m long, while the masses on the table and on the floor are 0·4 kg and 0·2 kg respectively. The particles are observed to have the same angular velocity. If the lower particle exerts zero force on the floor, find in terms of g:
(i) the tension in the string **(ii)** the speed of the particle on the table if the string were to break.

7. (a) (i) $P(x)$ is a polynomial with real coefficients. Show that if α is a non-real zero of $P(x)$, then $\overline{\alpha}$ is also a zero, and $(x-\alpha)(x-\overline{\alpha})$ is a quadratic factor of $P(x)$ with real coefficients.
(ii) Show that x^2+1 is a factor of $P(x) = x^6 + x^4 + x^2 + 1$, and hence factorise $P(x)$ into irreducible factors over the field of rational numbers.
(iii) Show that one solution of $x^4 + 1 = 0$ has argument $\frac{\pi}{4}$, and show the four complexd fourth roots of -1 on a unit circle on an Argand diagram.
(iv) Factorise $P(x) = x^6 + x^4 + x^2 + 1$ into irreducible factors over the field of real numbers.
(b) The equation $x^2 - x + 1 = 0$ has roots α and β. Let $A_n = \alpha^n + \beta^n,\ n = 1, 2, 3, \ldots$.
(i) Without solving the equation, show that $A_1 = 1$ and $A_2 = -1$.
(ii) Show that $A_n = A_{n-1} - A_{n-2},\ n = 3, 4, 5, \ldots$.
(iii) Use the method of mathematical induction to show that $A_n = 2\cos\frac{n\pi}{3},\ n = 1, 2, 3, \ldots$.

8. (a)



In the diagram above, ABC is a triangle inscribed in a circle. P is a point on the minor arc AB. L, M and N are the feet of the perpendiculars from P to CA (produced), AB, and BC respectively.

(i) Copy the diagram.

(ii) State a reason why P, M, A and L are concyclic points.

(iii) State a reason why P, B, N and M are concyclic points.

(iv) Show that L, M and N are collinear.

(b) Six lines are drawn in a plane. No two of the lines are parallel, and no three of the lines are concurrent.

(i) Show that there are 15 points of intersection.

(ii) If three of these points are chosen at random show that the probability that they all lie on one of the given lines is $\frac{12}{91}$.

(iii) Find the probability that if four of these points are chosen at random they do not all lie on one of the given lines.