CSSA of NSW

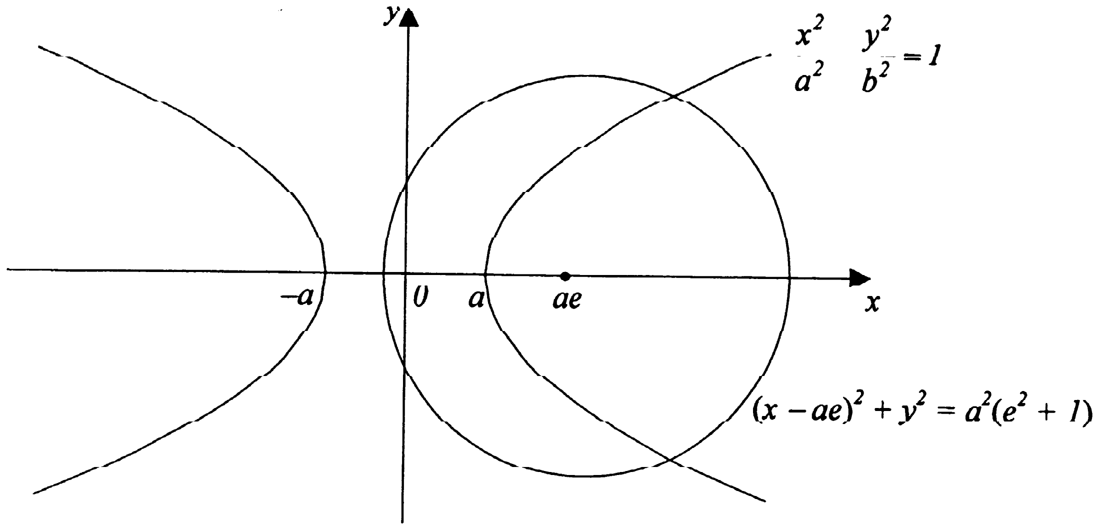
1993 Trial HSC 4 Unit Mathematics

1. (a) Sketch the graph of $y = \frac{x+3}{x+4}$ showing clearly the co-ordinates of any points of intersection with the x-axis and the y-axis, and the equation of any asymoptotes.
(b) Use the graph of $y = \frac{x+3}{x+4}$ in part **(a)** to find:
(i) the largest possible domain of the function $y = \sqrt{\frac{x+3}{x+4}}$.
(ii) the set of values of x for which the function $y = x - \log_e(x+4)$ is increasing.
(c) Use the graph of $y = \frac{x+3}{x+4}$ in part **(a)** to sketch on separate axes:
(i) the graph of $y = \left(\frac{x+3}{x+4}\right)^2$;
(ii) the graph of $y^2 = \frac{x+3}{x+4}$.
In each case state the nature of the point $(-3, 0)$.

2. (a) (i) Find $\int x\sec^2(x^2)\,dx$.
(ii) Find $\int \frac{x^4}{x^2+1}\,dx$.
(b) (i) Evaluate $\int_0^{\log_e 2} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$.
(ii) Use the substitution $u = \cos x$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^2 x}\,dx$.
(c) (i) Show that $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$.
(ii) If $I_n = \int_0^1 (1-\sqrt{x})^n\,dx$ for $n \geq 0$ show that $I_n = \frac{n}{n+2}I_{n-1}$ for $n \geq 1$.
(iii) Deduce that $\frac{1}{I_n} = \binom{n+2}{2}$ for $n \geq 0$.

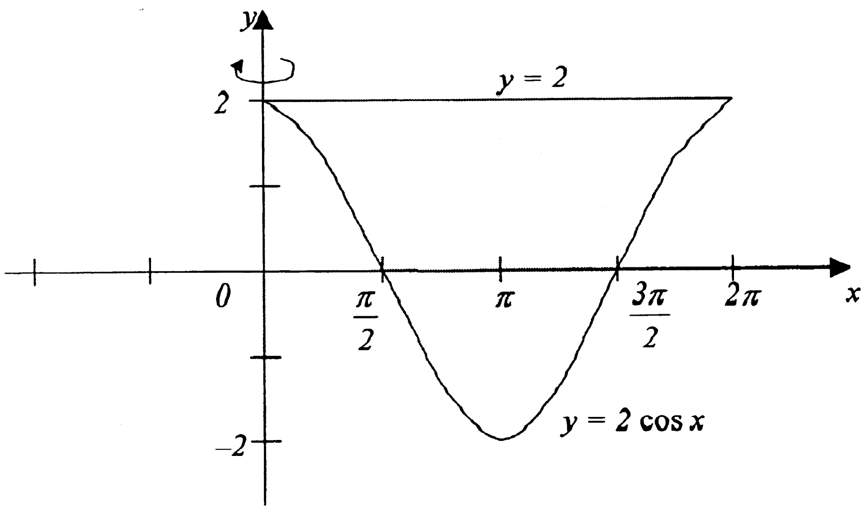
3. (a) (i) Show that $z = i$ is a root of the equation $(2-i)z^2 - (1+i)z + 1 = 0$.
(ii) Find the other root of the equation in the form $z = a + ib$, where a and b are real numbers.
(b) (i) Show that the locus specified by $|x-2| = 2\left(\Re(z) - \frac{1}{2}\right)$ is a branch of the hyperbola $\frac{x^2}{1} - \frac{y^2}{3} = 1$.
(ii) Sketch the locus, and find the set of possible values of each of $|z|$ and $\arg z$ for a point on the locus.
(c) z_1 and z_2 are two complex numbers such that $\frac{z_1+z_2}{z_1-z_2} = 2i$.
(i) On an Argand diagram show vectors representing z_1, z_2, z_1+z_2 and $z_1 - z_2$.
(ii) Show that $|z_1| = |z_2|$.
(iii) If α is the angle between the vectors representing z_1 and z_2 show that $\tan\frac{\alpha}{2} = \frac{1}{2}$.
(iv) Show that $z_2 = \frac{1}{5}(3+4i)z_1$.

4. (a)



(i) Show that the tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} - 1 = 0$.
(ii) Show that if the tangent at P is also a tangent to the circle with centre $(ae, 0)$ and radius $a\sqrt{e^2+1}$, then $\sec\theta = -e$.
(iii) Deduce that the points of contact P, Q on the hyperbola of the common tangents to the circle and hyperbola are the extremities of a latus rectum of the hyperbola, and state the coordinates of P and Q.
(iv) Find the equations of the common tangents to the circle and hyperbola, and find the coordinates of their points of contact with the circle.
(b) A sequence of numbers u_n is such that $u_1 = 3, u_2 = 21$, and $u_n = 7u_{n-1} - 10u_{n-2}$ for $n \geq 3$. Use the method of mathematical induction to show that $u_n = 5^n - 2^n$ for $n \geq 1$.

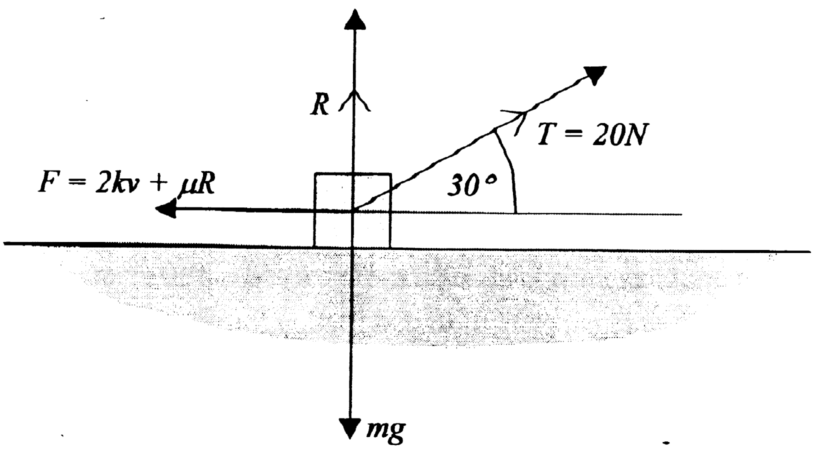
5. (a)



A mathematically inclined microwave cooking enthusiast decided to design his own

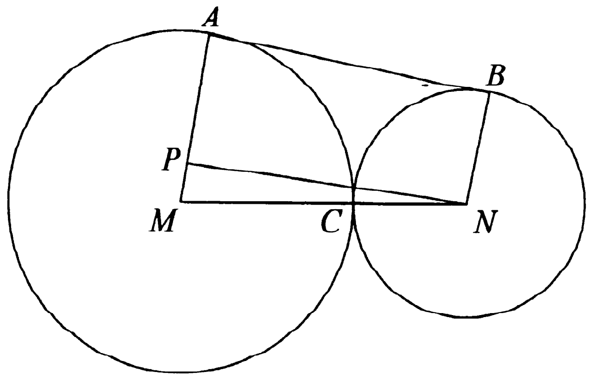
cake pan. The shape of the interior of the cake pan is obtained by rotating the region bounded by the curve $y = 2\cos x$, $0 \leq x \leq 2\pi$ and the line $y = 2$ through $360°$ about the y-axis. Use the method of cylindrical shells to show that the volume of the cake pan is given by $4\pi \int_0^{2\pi} x(1 - \cos x)\, dx$ and hence calculate this volume.
(b) The sequence $0 < x_1 < x_2 < x_3 < x_4 < \cdots$ comprises all the positive solutions of the equation $\tan x = x$.
(i) Use the graph of $y = \tan x$ to display graphically the first four members of the sentence.
(ii) by inspection of your graph, evaluate $\lim_{n\to\infty} (x_n - x_{n-1})$.
(iii) Show that if $\left(n\pi + \frac{\pi}{2}\right) - x_n < \lambda$, then $n > \frac{\cot\lambda}{\pi} - \frac{1}{2}$.
(iv) Hence find the number n of the first element x_n of the sequence which differs from an odd multiple of $\frac{\pi}{2}$ by less than $0{\cdot}01$.

6.

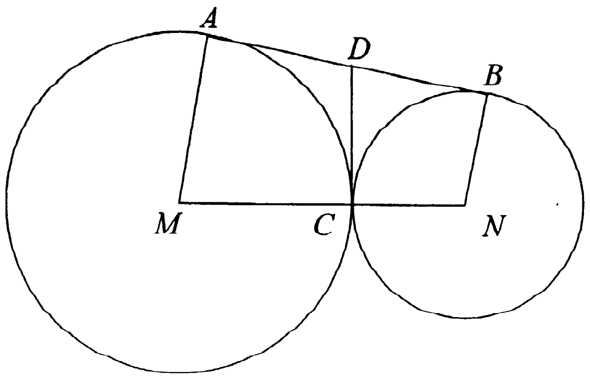


A mass of 2 kg slides on a horizontal surface, pulled by a rope inclined at $30°$ to the horizontal, with a constant tension in the rope of 20 N. Two resistance forces act horizontally on the body. One is a constant friction force of magnitude μR, where $\mu = 0{\cdot}2$ and R is the reaction force the surface exerts on the mass. The other resistance force has magnitude $2kv$, where k is a constant, and v is the speed of the body. Initially the body is travelling with half its terminal velocity. Take $g = 9{\cdot}8$ m.s^{-2}.
(i) Show that the equation of motion of the body is $\ddot{x} = b - kv$, where $s \approx 7{\cdot}7$.
(ii) Explain why this equation implies that the body has a terminal velocity of $\frac{b}{k}$.
(iii) Find as a function of v, the time t taken by the body to attain a velocity v.
(iv) The body is observed to attain 80% of its terminal velocity in 2 seconds. Find the value of k, and the distance travelled during these 2 seconds.

7. (a)



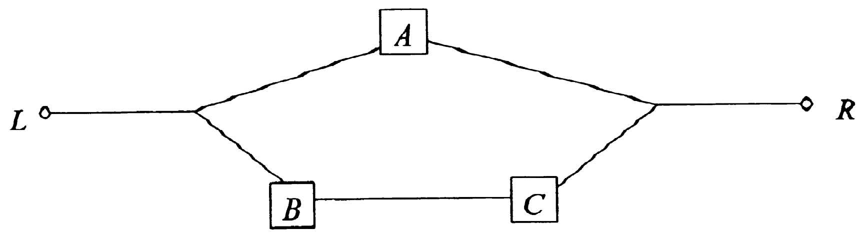
In the diagram MCN is a straight line with $MC = r$ and $NC = s$, where $r > s$. Circles are drawn with centre M, radius r and centre N, radius s. AB is a common tangent to the two circles with points of contact at A and B respectively. NP is drawn parallel to BA to meet MA at P. By considering $\triangle PMN$:
(i) show that $AB = 2\sqrt{rs}$;
(ii) show that $\sqrt{rs} < \frac{r+s}{2}$.
(b)



In the diagram MCN is a staight line. Circles are drawn with centre M, radius MC and centre N, radius NC. AB is a common tangent to the two circles with points of contact at A and B respectively. CD is the common tangent at C, and meets AB at D.
(i) Copy the diagram.
(ii) Explain why $AMCD$ and $BNCD$ are cyclic quadrilaterals.
(iii) Show that $\triangle ACD|||\triangle CBN$.
(iv) Show that $MD||CB$.

8. (a) (i) Write down the modulus and argument of each of i and $-i$.
(ii) Show on the unit circle on an Argand diagram the two square roots of i (z_1 and z_2), and the two square roots of $-i$ (z_3 and z_4).
(iii) $P(x) = x^4 + 1$. Show that the roots of $P(x) = 0$ are z_1, z_2, z_3 and z_4, and factor $P(x)$ completely over the real numbers.

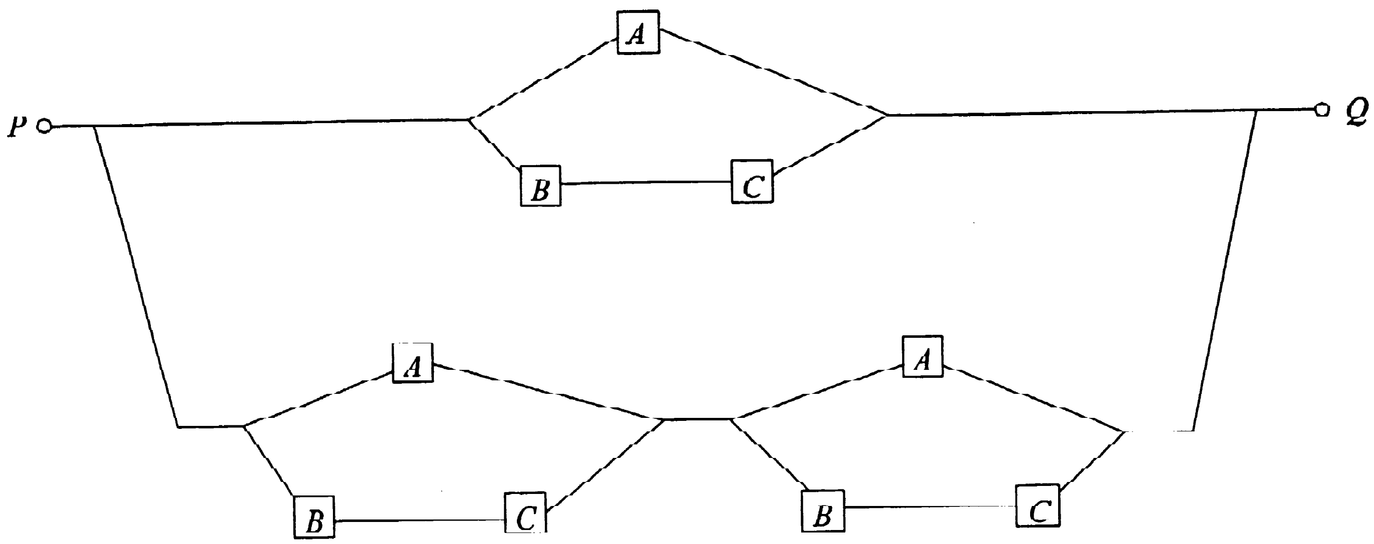
(b)



The diagram shows part of an electrical circuit. The boxes labelled A, B, C represent identical components, each of which, independently of the others has probability p of being defective. A current will not flow along any branch of the circuit in which there is a defective component.

(i) Show that the probability that a current cannot flow from L to R is $p^2(2-p)$.

(ii)



Find the probability that a current cannot flow from P to Q.