CSSA of NSW

1994 Trial HSC 4 Unit Mathematics

1. **(a)** Sketch the graph of $y = (2x+1)(x+1)$ showing clearly the intercepts on the coordinate axes and the coordinates of any turning points.
(b) **(i)** Use the graph in part **(a)** to sketch the graph of $y = \ln[(2x+1)(x+1)]$ showing the intercepts on the coordinate axes and the equations of any asymptotes.
(ii) Find the equation of the tangent to the curve $y = \ln[(2x+1)(x+1)]$ at the origin. Hence find the values of the real number k such that exactly one of the solutions of the equation $\ln[(2x+1)(x+1)] = kx$ is a positive number.
(c) **(i)** Use the graph in part **(a)** to sketch the graph of $y = \frac{1}{(2x+1)(x+1)}$ showing clearly the intercepts on the coordinate axes, the coordinates of any turning points and the equations of any asymptotes.
(ii) The region bounded by the curve $y = \frac{1}{(2x+1)(x+1)}$, the coordinate axes and the line $x = 4$ is rotated through one complete revolution about the y-axis. Use the method of cylindrical shells to find the volume of the solid generated.

2. **(a)** Find: **(i)** $\int \tan^2 x \, dx$ **(ii)** $\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx$.
(b) Evaluate $\int_1^6 x\sqrt{6-x} \, dx$.
(c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} \, dx$ using the substitution $t = \tan \frac{\pi}{2}$.
(d) Find: **(i)** $\int \ln x \, dx$ **(ii)** $\int 2x \ln(x^2+1) \, dx$.

3. **(a)** $-3+4i$ has two square roots z_1 and z_2. Find z_1 and z_2 in the form $a+ib$ and show the points representing $-3+4i, z_1$ and z_2 on an Argand diagram. Show that these three points are the vertices of a right angled triangle.
(b) The equation $z^2 + (1-2i)z - (7+i) = 0$ has roots α and β. Find the monic equation with numerical coefficients whose roots are $\alpha - i$ and $\beta - i$. Hence find the values of α and β.
(c) The complex number z is represented by the point P on an Argand diagram. Indicate clearly on a single diagram the locus of P in each of the following cases:
(i) $|z-4| = |z+2i|$; **(ii)** $\arg(z+3) = \frac{\pi}{4}$.
Show that there is a point representing a complex number of the form ib, where b is real, which lies on both loci.
(d) On an Argand diagram the point A represents the real number 1, O is the origin and the point P represents the complex number z which satisfies the condition $\arg(z-1) = 2\arg z$.
(i) Show this information on a diagram and deduce that triangle OAP is isosceles.
(ii) Deduce that the locus of P is a circle and show this circleon your diagram.
(iii) Find z in modulus/argument form if z also satisfies the condition $|z| = |z-1|$.

4. $P(20\cos\theta, 12\sin\theta)$ is a point on the ellipse $\frac{x^2}{20^2}+\frac{y^2}{12^2}=1$. P lies in the first quadrant, and the tangent to the ellipse at P meets the directrices in Q and R where Q is nearer the focus S' and R is nerer the focus S. Q and R each lie above the x-axis, and QS' meets RS in K where K lies in the third quadrant.
(i) Sketch the ellipse showing its directrices and foci and the points P, Q, R and K.
(ii) Show that the tangent at P has equation $3x\cos\theta + 5y\sin\theta = 60$.
(iii) Show that K has coordinates $\left(-20\cos\theta, \frac{4(9-25\sin^2\theta)}{3\sin\theta}\right)$.
(iv) If K lies on the ellipse, find the coordinates of P and show that $PSKS'$ is a rectangle.

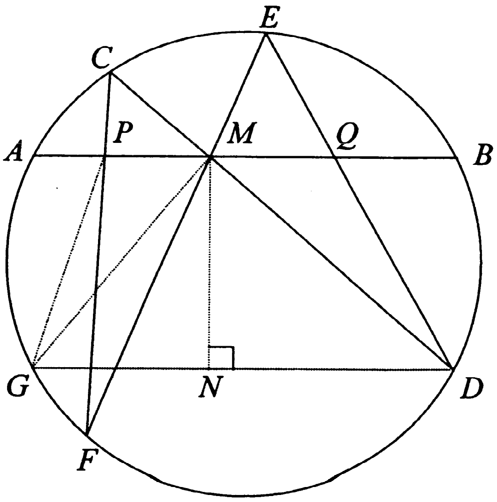
5. ABC is an isosceles triangle with $AB = BC = 2$ and $\angle ABC = \theta$. The circle with centre C and radius CA cuts AB internally at D and BC internally at E.
(i) Show this information on a sketch and show that $CA = 4\sin\frac{\theta}{2}$. State the maximum possible value of θ. **(ii)** Show that $\angle ACD = \theta$.
(iii) Show that if the chord AD and the arc DE are equal in length, then $\pi - 3\theta = 4\sin\frac{\theta}{2}$.
(iv) Use a graphical method to show there is exactly one value of θ for which the chord AD and the arc DE have equal length. Use Newton's Method to find this value of θ to the nearest $0{\cdot}1$ radians.

6. A particle of mass m is projected vertically upwards from a point high above the ground. The particle experiences a resistance of magnitude mkv^2 where k is a positive constant and the velocity of the particle has magnitude v. During its downward motion, the terminal velocity of the particle is V. Its initial velocity of projection is half this terminal velocity.
(i) By considering the forces on the particle during its downward motion, show that $kV^2 = g$.
(ii) Show that during its upward motion, the acceleration of the particle is given by $V^2\ddot{x} = -g(V^2+v^2)$, and the distance x travelled by the particle when its velocity is v is given by $x = \frac{V^2}{2g}\ln\left\{\frac{5V^2}{4(V^2+v^2)}\right\}$.
(iii) Find the maximum height h of the particle above its projection point.
(iv) Show that during it's downward motion, the acceleration of the particle is given by $V^2\ddot{x} = g(V^2-v^2)$.
(v) Find the position of the particle relative to its projection point when it attains 60% of its terminal velocity.

7. (a) (i) Show that $\tan(A+\frac{\pi}{2}) = -\cot A$.
(ii) Use the method of mathematical induction to show that $\tan\left\{(2n+1)\frac{\pi}{4}\right\} = (-1)^n$ for all integers $n \geq 1$.
(b) $P(x) = x^6 + x^3 + 1$.
(i) Show that the roots of $P(x) = 0$ are amongst the roots of $x^9 - 1 = 0$.
(ii) Hence show the roots of $P(x) = 0$ on the unit circle, centre the origin, on an Argand diagram.

(iii) Show that $P(x) = (x^2 - 2x\cos\frac{2\pi}{9} + 1)(x^2 - 2x\cos\frac{4\pi}{9} + 1)(x^2 - 2x\cos\frac{8\pi}{9} + 1)$.
(iv) Evaluate $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} + \cos\frac{4\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{8\pi}{9}\cos\frac{2\pi}{9}$.

8. **(a)** The quadratic equation $x^2 - x + k = 0$, where k is a real number, has two distinct positive real roots α and β.
(i) Show that $0 < k < \frac{1}{4}$.
(ii) Show that $\alpha^2 + \beta^2 = 1 - 2k$ and deduce that $\alpha^2 + \beta^2 > \frac{1}{2}$.
(iii) Show that $\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$.
(b)



Chords AB, CD, EF are concurrent in M where M is the midpoint of AB. CF, ED meet AB in P, Q respectively. Chord DG is constructed parallel to AB, and N is the foot of the perpendicular from M to DG.
(i) Copy the diagram.
(ii) Show that $\triangle MGD$ is isosceles.
(iii) Show that $PMFG$ is a cyclic quadrilateral.
(iv) Show that $MP = MQ$.