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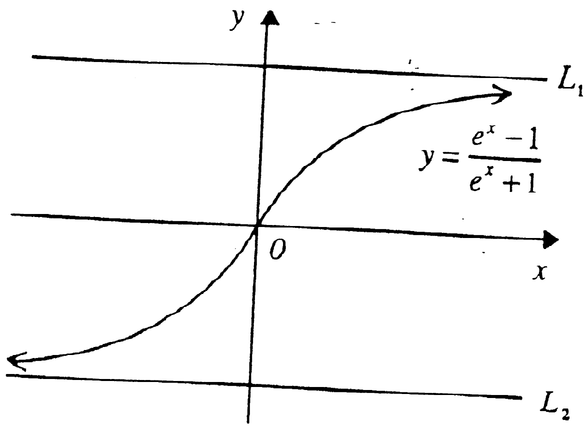
1. Consider the function $f(x) = \frac{e^x-1}{e^x+1}$.
(a) (i) Show that the function is odd.
(ii) Show that the function is always increasing.
(b)



This diagram shows the graph of $y = \frac{e^x-1}{e^x+1}$, where $(0,0)$ is a point of inflexion. Copy the diagram. Find the equations of the asymptotes L_1 and L_2 and show these on your diagram.
(c) (i) Find the equation of the tangent to the curve $y = \frac{e^x-1}{e^x+1}$ at $(0,0)$.
(ii) Find the values of k for which the equation $\frac{e^x-1}{e^x+1} = kx$ has 3 real solutions.
(d) Without using calculus, use the graph of $y = \frac{e^x-1}{e^x+1}$ to sketch on separate axes, the graphs **(i)** $y = \frac{e^x+1}{e^x-1}$ **(ii)** $y = \left(\frac{e^x-1}{e^x+1}\right)^2$.

2. (a) (i) Show $\int_1^2 \frac{1}{x^2} \ln(x+1)\, dx = \frac{1}{2} \ln \frac{4}{3} + \int_1^2 \frac{1}{x(x+1)}\, dx$.
(ii) Hence evaluate $\int_1^2 \frac{1}{x^2} \ln(x+1)\, dx$ in simplest exact form.
(b) (i) Show that $\int_0^1 x(1-x)^n\, dx = \frac{n!}{(n+2)!}$.
(ii) Evaluate $\int_0^{\frac{\pi}{2}} \sin 2x(1-\sin x)^{10}\, dx$ using the substitution $u = \sin x$.
(c) (i) If $\theta = \pi - \tan^{-1} \frac{3}{4}$, show $\tan \frac{\theta}{2} = 3$.
(ii) Evaluate $\int_{\frac{\pi}{2}}^{\pi - \tan^{-1}\frac{3}{4}} \frac{1}{\cos\theta + 2}\, d\theta$, leaving your answer in terms of π.

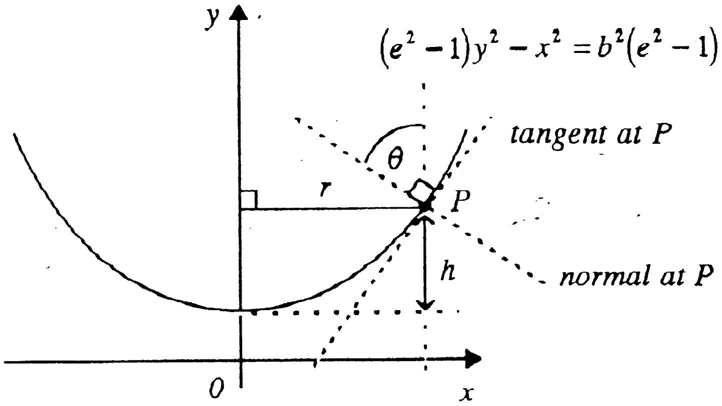
3. (a) (i) If $z = \cos\theta + i\sin\theta$, show that $1 + z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$.
(ii) z_1, z_2 are complex numbers such that $|z_1| = |z_2| = 1$. If z_1, z_2 have arguments α, β respectively, where $-\pi < \alpha \le \pi$ and $-\pi < \beta \le \pi$, show that $\frac{z_1+z_1z_2}{z_1+1}$ has modulus $\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$ and argument $\frac{\alpha+\beta}{2}$.
(iii) $|z_1| = |z_2| = 1$ and $\frac{z_1+z_1z_2}{z_1+1} = 2i$. Find z_1 and z_2 in the form $x + iy$.
(b) On an Argand diagram the points P, Q, R represent the complex numbers p, q, r respectively. If $p - q + iq - ir = 0$, what type of triangle is $\triangle PQR$? Give a reason

for your answer.
(c) (i) On an Argand diagram sketch the locus of the point P representing the complex number z which moves so that $z - 2| = 1$.
(ii) Find the sets of possible values of $|z|$ and $\arg z$.
(iii) The points P_1 and P_2 such that OP_1 and OP_2 are tangents to the locus, represent the complex numbers z_1, z_2 respectively. Express each of z_1 and z_2 in modulus/argument form and in the form $a + ib$, where a and b are real.
(iv) Evaluate $z_1^{20} + z_2^{20}$.

4. The hyperbola $xy = c^2$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ where $t_1 > t_2 > 0$. Tangents to the hyperbola at P and Q meet in T, while tangents to the ellipse at P and Q meet in V.
(i) Show this information on a sketch.
(ii) Show that the parameter t of a point $\left(ct, \frac{c}{t}\right)$ where $xy = c^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect satisfies the equation $b^2c^2t^4 - a^2b^2t^2 + a^2c^2 = 0$.
(iii) Using without proof the result that the tangent to hyperbola $xy = c^2$ at the point $\left(ct, \frac{c}{t}\right)$ has equation $x + t^2y = 2ct$, show that T has coordinates $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.
(iv) Using without proof the result that the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) has equation $b^2x_1x + a^2y_1y = a^2b^2$, show that V has coordinates $\left(\frac{a^2}{c(t_1+t_2)}, \frac{b^2t_1t_2}{c(t_1+t_2)}\right)$.
(v) Show that TV passes through the origin.
(vi) Show that if V lies at a focus of the hyperbola, then the ellipse is a circle and find the radius of this circle in terms of c.

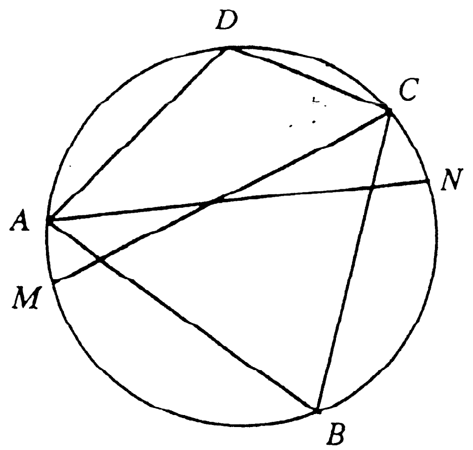
5. (a) (i) If α is a double zero of the polynomial $P(x)$, show that α is a zero of $P'(x)$.
(ii) $(x - 1)^2$ is a factor of $x^5 + 2x^4 + ax^3 + bx^2$. Find the values of a and b.
(b) $\sqrt{3} + i$ is one root of $x^4 + px^2 + q = 0$, where p and q are real. Find p and q and factor $x^4 + px^2 + q$ into quadratic factors with real coefficients.
(c) The region bounded by the curve $y = e^x$, the y-axis and the line $y = 2$ is rotated through $360°$ about the y-axis. Use the method of cylindrical shells (taking strips parallel to the y-axis) to find the volume of the solid of revolution.

6.



A bowl is made by rotating the branch of the hyperbola shown above about the y-axis. A particle P of mass m travels around the bowl in a horizontal circle of radius r with constant speed v at a height h above the bottom of the bowl.
(a) **(i)** Write down the coordinates of P and deduce that $r^2 = \{(h+b)^2 - b^2\}(e^2-1)$.
(ii) If the normal to the hyperbola at P makes an angle θ with the vertical, show that $\tan\theta = \frac{r}{(e^2-1)(h+b)}$.
(b) If the particle P has no tendency to slip up or down the bowl, and the bowl is smooth,
(i) draw a diagram showing the forces acting on P.
(ii) show $v = v_0$ where $v_0^2 = \frac{g}{h+b}\{(h+b)^2 - b^2\}$. Will this linear speed v_0 increase or decrease with increasing height h of P up the bowl? Will the corresponding angular velocity ω_0 increase or decrease with increasing h? Give reasons for your answers.
(iii) Show that when $v = v_0$, the force the particle exerts on the bowl has magnitude $\frac{mg}{\sqrt{e^2-1}}\sqrt{e^2 - \left(\frac{b}{h+b}\right)^2}$.
(c) If the bowl were not smooth so that friction forces acted on the particle, slowing down the linear speed v, describe *qualitatively* what you would expect to see and any changes in the force exerted by the particle on the bowl.

7. (a)



In the diagram $ABCD$ is a cyclic quadrilateral. M and N are points on the circle through A, B, C and D such that CM bisects $B\hat{C}D$ and AN bisects $D\hat{A}B$.
(i) Copy the diagram.
(ii) Show that MN is a diameter of the circle.
(b) The numbers a, b, c are said to be in harmonic progression if their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and b is then said to be the harmonic mean of a and c.
(i) Show that the numbers 6,8,12 are in harmonic progression.
(ii) Show that the harmonic mean of a and c is equal to $\frac{2ac}{a+c}$.
(iii) If $a > 0, c > 0$ show that the geometric mean $\sqrt{ac}$ is greater than or equal to the harmonic mean $\frac{2ac}{a+c}$.

8. (a) A sequence of numbers $T_n, n = 1, 2, 3, \ldots$ is given by $T_1 = 1, T_2 = 3$ and $T_n = T_{n-1} + T_{n-2}, n = 3, 4, 5, \ldots$. Use the method of mathematical induction to show that $T_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, n = 1, 2, 3, \ldots$.
(b) Team A plays team B in a chess match. Each team consists of four players. The match consists of four games where each member of one team is paired with exactly one member of the other team. Each game has three possible outcomes:

a win for the player from team A	(1 point for A, 0 points for B)
a win for the player from team B	(1 point for B, 0 points for A)
a draw	($\frac{1}{2}$ point for B, $\frac{1}{2}$ point for A)

(i) Find the number of different ways in which the pairings can be made between the players of team A and team B.
(ii) For a particular arrangement of pairs of players, find the number of different possible ways in which the outcomes of the four games can occur.
(iii) Suppose that the players in team A and team B are evenly paired so that in each of the four games, the outcomes 'a win for A', 'a win for B' and 'a draw' are equally likely. For this particular pairing, find the probability that the overall match is drawn (i.e., a total of 2 points for each team).