CSSA of NSW

1996 Trial HSC 4 Unit Mathematics

1. **(a)** A function $y = f(x)$ is given in parametric form by

$$\left.\begin{aligned} x &= \tan\theta \\ y &= 4\sin 2\theta \end{aligned}\right\}, -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}.$$

(i) State the domain and range of the function and show that $y = \frac{8x}{1+x^2}$.
(ii) Sketch the graph of $y = f(x)$ showing clearly the coordinates and nature of any stationary points and the equations of any asymptotes.
(iii) Use the graph of $y = f(x)$ to sketch on separate axes the graphs of:
(α) $y = \{f(x)\}^2$;
(β) $y^2 = f(x)$.
(b) If $y = \tan^{-1}(e^x)$ show that $\frac{dy}{dx} = \frac{1}{2}\sin 2y$.

2. **(a)** Find $\int \cos^2 x \; dx$.
(b) Simplify $\frac{1}{1-\sin x} - \frac{1}{1+\sin x}$ and hence find $\int \left(\frac{1}{1-\sin x} - \frac{1}{1+\sin x}\right) dx$.
(c) Use the substitution $x = u^2, u > 0$, to evaluate $\int_1^9 \frac{1}{x(\sqrt{x}+1)} \; dx$.
(d) $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}} \; dx, \;\; n = 0, 1, 2, 3, \ldots$.
(i) Show that $x^{n-1}\sqrt{x+1} = \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}}$.
(ii) Show that $(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}, \;\; n = 1, 2, 3, \ldots$.
(iii) Evaluate $\int_0^1 \frac{x^3}{\sqrt{x+1}} \; dx$.

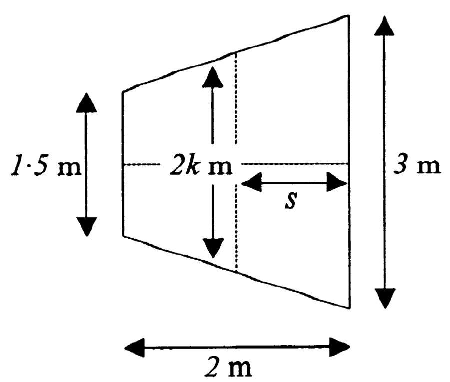
3. **(a)** $z_1 = 1 + 3i, z_2 = 1 - i$
(i) Find in the form $a + ib$, where a and b are real, the numbers $z_1 z_2$ and $\frac{z_1}{z_2}$.
(ii) On an Argand diagram the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ represent the complex numbers $z_1 z_2$ and $\frac{z_1}{z_2}$ respectively (where z_1 and z_2 are given above). Show this on an Argand diagram, giving the coordinates of A and B. From your diagram, deduce that $\frac{z_1}{z_2} - z_1 z_2$ is real.
(b) The complex number β is given by $\beta = (1 - t^2) + (2t)i$, where t is real.
(i) Show that $|\beta| = 1 + t^2$ and $\arg\beta = \theta$ where $t = \tan\left(\frac{\theta}{2}\right)$.
(ii) Write down the modulus and argument of one square root of β, and hence if $z^2 = \beta$, write z in the form $a + ib$ where a and b are real.
(iii) Hence or otherwise find the 2 square roots of $-8 + 6i$.
(c) $z = \cos\theta + i\sin\theta, -\pi < \theta < \pi$
(i) Show that $1 + z = 2\cos\left(\frac{\theta}{2}\right)\left\{\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right\}$.
(ii) Show that $1 - z = 2\sin\left(\frac{\theta}{2}\right)\left\{\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right\}$.
(iii) Show that $\frac{1-z}{1+z} = -i\tan\left(\frac{\theta}{2}\right)$.

(iv) Sketch the locus of z if $|z| = 1$ and $\left|\frac{1-z}{1+z}\right| \le \frac{1}{\sqrt{3}}$. Find z on this locus if $\Im(z)$ takes its maximum value.

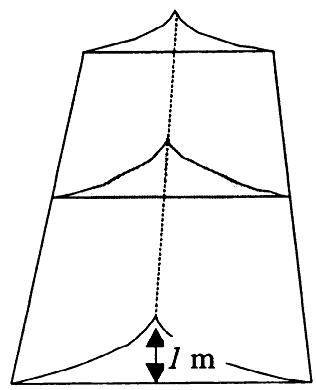
4. **(a)** The diameter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b > 0$) through the point $P(a\cos\theta, b\sin\theta)$ meets the circle $x^2 + y^2 = a^2$ at the point $R(a\cos\phi, a\sin\phi)$.
(i) Show this information on a sketch.
(ii) Show that $\tan\phi = \frac{b}{a}\tan\theta$.
(iii) Prove that the tangent to the ellipse at P has equation $bx + ay\tan\theta = ab\sec\theta$.
(iv) Show that the tangent to the circle at R has equation $ax + by\tan\theta = a^2\sec\phi$.
(v) If the tangent to the ellipse at P and the tangent to the circle at R are concurrent with the right hand directrix of the ellipse, show that $\sec\theta = \frac{2}{e}$, where e is the eccentricity of the ellipse.
(b) The diameter of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ through the point P on the ellipse meets the circle $x^2 + y^2 = 25$ at R. Tangents to the ellipse at P and the circle at R are concurrent with a directrix of the ellipse. Using the results from part **(a)**: If P lies in the first quadrant,
(i) find the coordinates of P and R.
(ii) Sketch the ellipse and the circle, showing the coordinates of P and R, and the point of intersection of the tangents and the appropriate directrix.

5. **(a)** The region R is bounded by the x-axis and the curve $|x|^{\frac{1}{2}} + (ky)^{\frac{1}{2}} = k^{\frac{1}{2}}$, where k is a positive constant.
(i) Show that $x^{\frac{1}{2}} + (ky)^{\frac{1}{2}} = k^{\frac{1}{2}}, k > 0$, defines y to be a monotonic decreasing function of x throughout its domain. Comment on the gradient of the tangent at any critical point on the curve.
(ii) Sketch the graph of the curve $x^{\frac{1}{2}} + (ky)^{\frac{1}{2}} = k^{\frac{1}{2}}, k > 0$, and hence on the same diagram sketch the graph of the curve $|x|^{\frac{1}{2}} + (ky)^{\frac{1}{2}} = k^{\frac{1}{2}}$.
(iii) Shade the region R. Show that the region R has area $\frac{1}{3}k$ square units.
(b)

Trapezium base of the tent.



Tent showing typical cross-section.



The base of a tent is a trapezium with parallel sides of length 1·5 metres at the back of the tent and 3 metres at the front of the tent. The base has an axis of symmetry perpendicular to the parallel sides and is 2 metres long. The roof of the tent is

formed by draping material over a horizontal ridge pole of length 2 metres directly above the axis of symmetry of the base and at a height of 1 metre, as shown in the diagram above. Vertical cross-sections taken perpendicular to the axis of symmetry of the base have the shape of the region R described in part **(a)**, where $2k$ metres is the width of the cross section where it meets the trapezium base.
(i) Show that if at a distance s metres from the front of the tent (measured along the axis of symmetry of the trapezium) the width of the trapezium base is $2k$ metres, as shown in the diagram, then $k = \frac{3}{2}\left(1 - \frac{1}{4}s\right)$.
(ii) Deduce that the area of a typical cross-section as shaded above, taken at a distance s metres from the front of the tent, is $\frac{1}{2}\left(1 - \frac{1}{4}s\right)$ square units.
(iii) If the tent has vertical flaps front and back, calculate the volume of the interior of the tent.

6. A particle of mass m is projected vertically upwards under gravity. The air resistance to the motion is $\frac{1}{100}mgv^2$ where v is the speed of the particle.
(i) Show that during the upward motion of the particle, if x is the upward vertical displacement of the particle from its projection point at time t, then $\ddot{x} = -\frac{1}{100}g(100 + v^2)$. If the speed of projection is u, show that the greatest height (above the projection point) reached by the particle is $\frac{50}{g}\ln\left(\frac{100+u^2}{100}\right)$.
(ii) Show that during the downward motion of the particle, if x is the downward vertical displacement of the particle from its highest point at a time t after it begins the downward motion, then $\ddot{x} = \frac{1}{100}g(100 - v^2)$. Show that the speed of the particle on return to its point of projection is $\frac{10u}{\sqrt{100+u^2}}$.
(iii) Find the terminal velocity V of the particle for the downward motion. If the initial speed of projection of the particle is V, show that the speed on return to the point of projection is $\frac{1}{\sqrt{2}}V$.

7. **(a)** The quadratic equation $x^2 - (2\cos\theta)x + 1 = 0$ has roots α and β.
(i) Find expressions for α and β.
(ii) Show that $\alpha^{10} + \beta^{10} = 2\cos(10\theta)$.
(b) In an Argand diagram, A, B, C, D represent the complex numbers $\alpha, \beta, \gamma, \delta$ respectively.
(i) Describe the point which represents $\frac{1}{2}(\alpha + \gamma)$.
(ii) Deduce that if $\alpha + \gamma = \beta + \delta$, then $ABCD$ is a parallelogram.
(c) $x^4 - 2(1+\beta)x^3 + \frac{3}{2}(1+\beta)^2x^2 - \frac{1}{2}(1+\beta)^3x + \frac{1}{2}\beta(1+\beta^2) = 0$.
(i) Show that 1 and β are roots of this equation.
(ii) If the other roots are α and γ, use expressions for the sum and product of the roots to show that $\alpha + \gamma = 1 + \beta$ and $\alpha\gamma = \frac{1}{2}(1 + \beta^2)$. Hence show that $\alpha - \gamma = \pm i(1 - \beta)$.
(iii) Deduce that the points representing $1, \alpha, \beta, \gamma$ in the Argand diagram are the vertices of a square.
(iv) Hence or otherwise find the four roots of the equation

$$x^4 - 2(2+i)x^3 + \frac{3}{2}(2+i)^2x^2 - \frac{1}{2}(2+i)^3x + \frac{1}{2}(1+i)(1+2i) = 0$$

and show the points representing these roots on an Argand diagram.

8. n letters $L_1, L_2, L_3, \ldots, L_n$ are to be placed at random into n addressed envelopes $E_1, E_2, E_3, \ldots, E_n$, each bearing a different address, where E_i bears the correct address for letter L_i, $i = 1, 2, 3, \ldots, n$. Let u_n be the number of arrangements where no letter is placed in the correct envelope, for n a positive integer, $n \geq 2$.
(i) Show $u_2 = 1$ and $u_3 = 2$.
(ii) Deduce that $u_{k+1} = k(u_k + u_{k-1})$, $k = 4, 5, 6, \ldots$.
(iii) Use the results from **(i)** and **(ii)** to calculate u_4 and u_5.
(iv) Show by mathematical induction that $u_n = n!\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}\}$, $n = 2, 3, 4, \ldots$.
(v) If there are 5 letters and envelopes:
($\boldsymbol{\alpha}$) Explain why the probability no letter is placed in the correct envelope is $\frac{u_5}{120}$ and calculate this probability as a fraction;
($\boldsymbol{\beta}$) Show the probability that exactly one letter is placed in the correct envelope is $\frac{5u_4}{120}$ and calculate this probability as a fraction.
($\boldsymbol{\gamma}$) Find the probability exactly 2 letters are placed in the correct envelopes.
(vi) Deduce that $\sum\limits_{k=2}^{n} \binom{n}{k} u_k = n! - 1$.