CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1997

MATHEMATICS

4 UNIT COURSE

Morning session

Wednesday, 6th August 1997

Examiners

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DIRECTIONS TO CANDIDATES :

- All questions may be attempted.
- All questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Standard integrals are printed on a separate page. Approved calculators may be used.

(a) Consider the curve $y=\cos\left(\sqrt{x}\right),\ 0\le x<4\pi^2$

dx

(ii) Find any stationary points and determine their nature.

(iii) Sketch $y=\cos\left(\sqrt{x}\right),\ 0\le x\le 4\pi^2$ then complete the sketch to show the

coordinate axes, the limiting tangents at any critical points, and the coordinates of any stationary points.

(iv) [illegible]

(b) A curve has parametric equations $\left.\begin{array}{l} x=\sin\theta \\ y=\tan\theta \end{array}\right\}$, $-\pi<\theta\le\pi\ ,\ \theta\ne\pm\frac{\pi}{2}$ **3**

Question 2

(b) Use the substitution $t=\tan\frac{\theta}{2}$ to find $\int \frac{1}{1-\cos\theta-\sin\theta}\,d\theta$ **4**

(c) **3**

(i) Use the substitution $u=\frac{\pi}{2}-x$ to show that $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx = \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\sin x}+e^{\cos x}}\,dx$

(ii) Hence evaluate $\int^{\pi} \frac{e^{\sin x}}{[illegible]}\,dx$

(d) $I_n=\int_0^1 \frac{x^n}{x^2+1}\,dx$, $n\ge 0$, n integral. **5**

(i) Show that $I_n + I$ [illegible] $= \frac{1}{[illegible]}$

(ii) Evaluate I_4 and I_5

Marks

Question 3

(a) $\beta = 2 + 3i$ **5**

(i) On an Argand Diagram show the point B representing β , and vectors representing $\beta - 1$ and , $\beta - i$.

(ii) If θ is the acute angle between vectors $\beta - 1$ and $\beta - i$, show that $\tan\theta = \frac{1}{2}$.

$|\beta - 1|^2$ 5

(iv) Find $\left(\dfrac{\beta - 1}{\beta - i}\right)^2$ in the form $a + ib$, a, b real.

(b) $x^2 - 2x + p = 0$, p real and $p > 1$, has roots α and β . **5**

(i) Show that α and β are non-real.

(ii) Show the relative positions of points A and B representing α and β on an Argand Diagram.

(c) z satisfies $|z - 2i| = 1$, and the point P represents z on an Argand Diagram. **5**

(i) Sketch the locus of P as z varies.

(ii) Find the maximum and minimum values of $\arg z$, where $-\pi < \arg z \leq \pi$.

(iii) Find the value of z when $\arg z$ takes this minimum value, and mark on your sketch the position P_0 of P for this value of z.

Question 4

Marks

(a) Tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ intersect at T. M is the midpoint of PQ **9**

write down the equation of the tangent to the ellipse at Q.

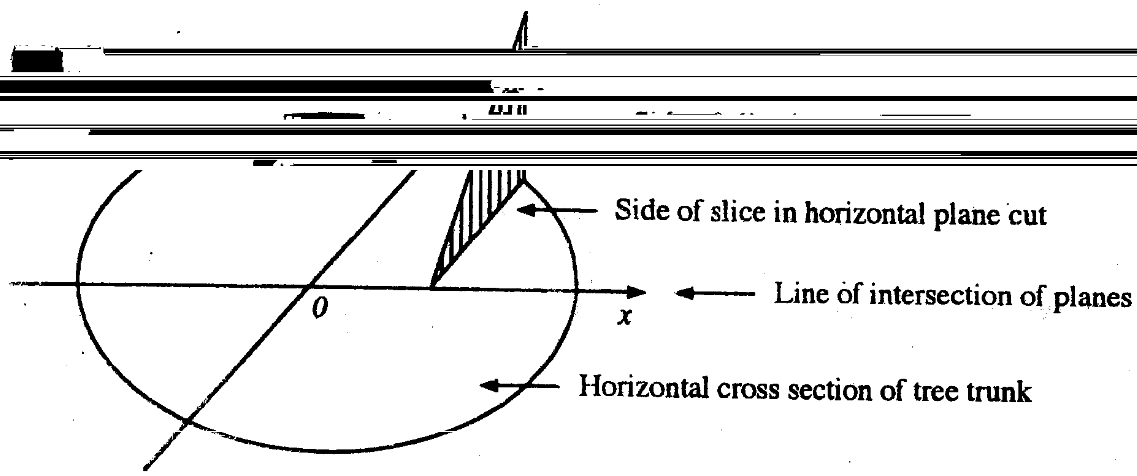
(ii) Show that the line $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{xx_2}{a^2} + \frac{yy_2}{b^2}$ passes through T and M.

(iii) Deduce that the points O, T, M are collinear.

(iv) Show that the product of the gradients of PQ and TM is a constant.

(b) A vertical tree, circular in cross section with a diameter of 60cm, has a notch cut out by two planes. One plane is horizontal, while the second plane makes an angle of 45° with the first. The two planes meet along a diameter of a circular cross section. **6**

The volume V of the notch is obtained by taking slices perpendicular to the diameter in



(i) Show that the volume of the notch is given by $V = \int^{30} \left(900 - x^2\right) dx$

Marks

Question 5

(a) 8

(i) Use De Moivre's Theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(ii) Deduce $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$ where $\cos 3\theta = \frac{1}{2}$.

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $\cos\theta$.

(iv) Hence evaluate $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$.

(b) $f(x) = x^2(x^2 - 2)$. The tangent to the curve $y = f(x)$ at the point A with x coordinate α meets the curve again at B. 7

(i) Show the tangent AB has equation $y = 4\alpha(\alpha^2 - 1)x + \alpha^2(2 - 3\alpha^2)$.

(ii) Deduce that $x^2(x^2 - 2) = 4\alpha(\alpha^2 - 1)x + \alpha^2(2 - 3\alpha^2)$ has real roots $\alpha, \alpha, \beta, \gamma$ for some β, γ.

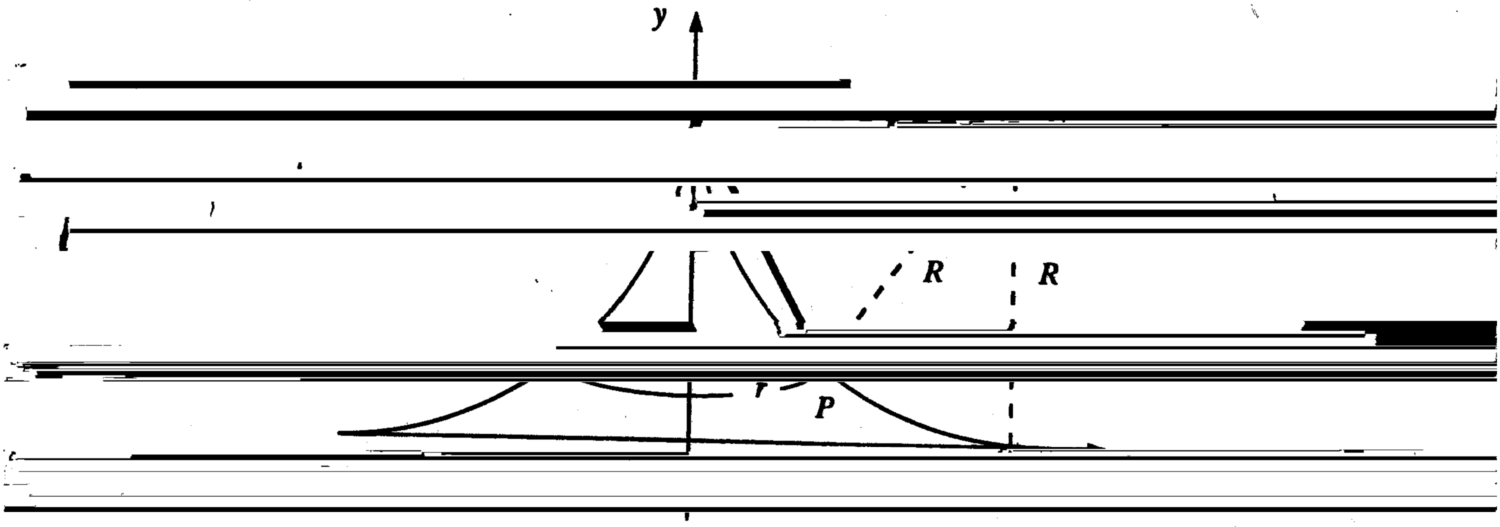
(iii) For $\alpha \neq 0$, find $\beta + \gamma$ and $\beta\gamma$ in terms of α and write down a quadratic equation with roots β, γ .

(iv) Find the possible values of α .

Question 6

Marks

15



AB is an arc of a circle centre C and radius R. A surface is formed by rotating the arc AB through one revolution about the y axis. A light, inextensible string of length l, $l \leq R$, is attached to point A, and a particle of mass m is attached to the other end. The particle is set in motion, tracing out a horizontal circle on the surface with constant angular velocity ω radians per second, while the string stays taut.

(i) When the particle is in the position P shown on the diagram, explain why the direction of the force N exerted by the surface on the particle is towards C.

(ii) If the string makes an angle θ with the vertical, show that $\angle ACP = 2\theta$.

(iii) Show on a diagram the tension force T, the force N and the weight force of magnitude

(iv) Show that

$$T\cos\theta + N\sin 2\theta = mg$$
$$T\sin\theta - N\cos 2\theta = m\,l\sin\theta\,\omega^2$$

(v) Show that $N = m\,l\sin\theta\left(\frac{g}{l}\sec\theta - \omega^2\right)$

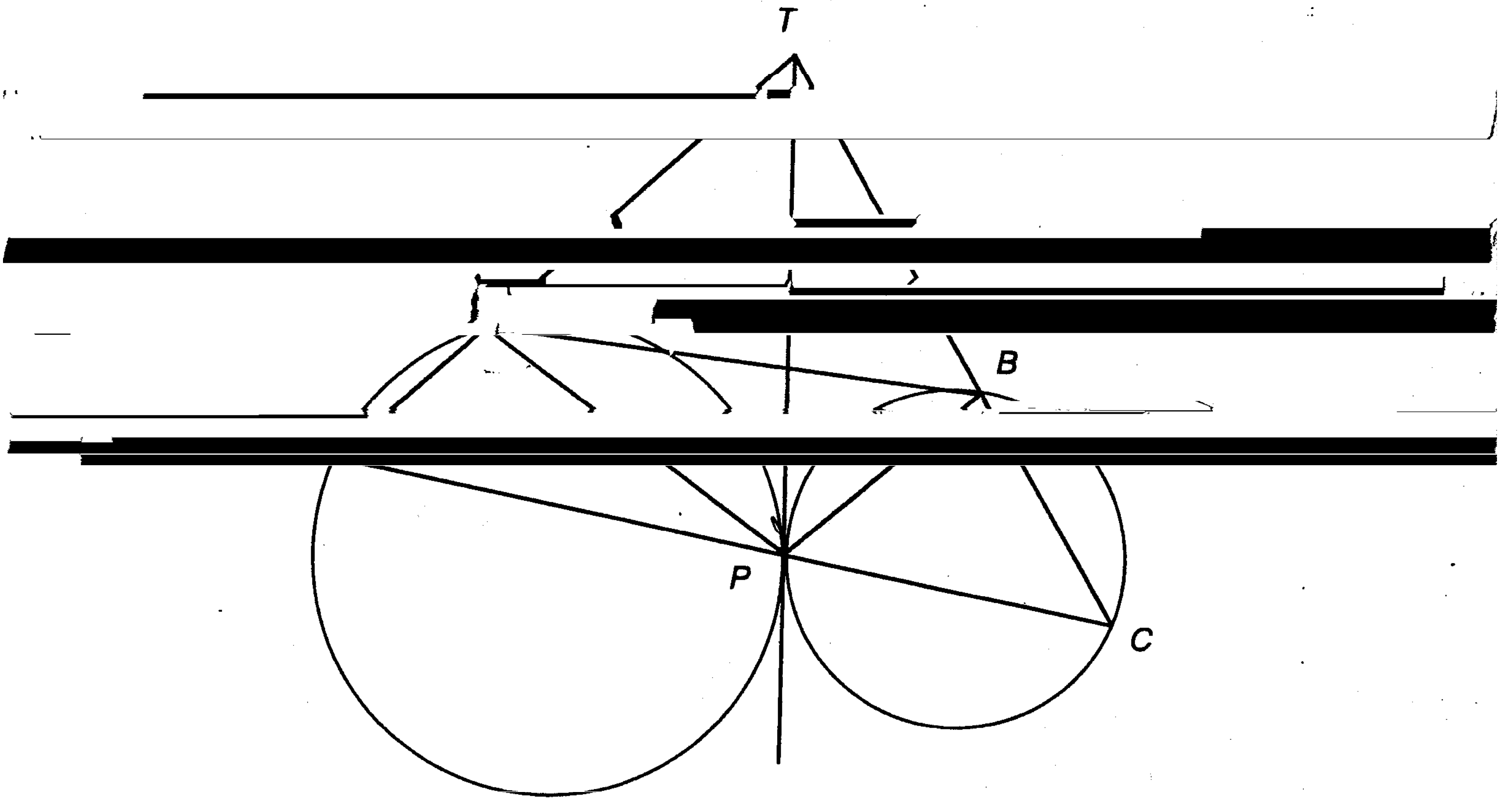
(vi) Deduce that there is a maximum value ω for the motion to occur as described, and write down this maximum value. What happens if ω exceeds this maximum ?

(vii) If $l = R$, find T in terms of l, m and ω^2. Describe what happens to the tension in the string and the force the particle exerts on the surface as ω increases.

Question 7

Marks

(a) 7



Circles APD and BPC touch at P. D, P and C are collinear. TP is the common tangent at P. TC cuts circle BPC in B while TD cuts circle APD in A.

(i) Copy the diagram.

(ii) Show that $ATBP$ is a cyclic quadrilateral.

(iii) Show that $ABCD$ is a cyclic quadrilateral.

(b) 8

(ii) Use the method of mathematical induction to show that for all positive integers $n \geq 2$, if $x_j > 1$, $j = 1, 2, 3, \ldots, n$ then

$$\ln(x_1 + x_2 + \ldots + x_n) > \frac{1}{2^{n-1}}(\ln x_1 + \ln x_2 + \ldots + \ln x_n)$$

Marks

Question 8

(a) **3**

(i) Show that for $a > 0$ and $n \neq 0$, $\log_{a^n} x = \frac{1}{n} \log_a x$.

[redacted]

(b) **12**

(i) Show that Binomial coefficients $^nC_{r-1}$, ^nC_r are related by the equation

$$^nC_r = \frac{n-r+1}{r}\ ^nC_{r-1} .$$

Suppose $^nC_{[illegible]}$, $^nC_{[illegible]}$, $^nC_{[illegible]}$ are three consecutive terms of an arithmetic progression.

[redacted]

[redacted]

(iii) If $n+2 = p^2$, show that either $2(r+1) = p(p+1)$ or $2(r+1) = p(p-1)$.

[redacted]

of Pascal's triangle, evaluate the consecutive binomial coefficients which are in arithmetic progression.

(v) Deduce that no row of Pascal's triangle contains a set of four consecutive elements which are in arithmetic progression.