CSSA of NSW

1998 Trial HSC 4 Unit Mathematics

1. (a) Consider the function $f(x) = \frac{x-1}{x+3}$.
(i) Sketch the graph of $y = f(x)$ showing clearly the coordinates of any point of intersection with the x-axis or the y-axis, and the equations of any asymptotes.
(ii) Show that the line $y = x$ is a tangent to the curve $y = f(x)$ and find the coordinates of its point of contact. Draw the tangent line on the graph and show the coordinates of its point of contact.
(iii) On separate axes, sketch the graphs of $y = \frac{1}{f(x)}$ and $y = f^{-1}(x)$. In each case, show clearly the coordinates of any points of intersection with the x-axis or the y-axis, the equations of any asymptotes, and the line $y = x$.
(b) (i) On the same diagram sketch the graphs of $x^2 + y^2 = 9$ and $x^2 - y^2 = 4$ showing clearly the coordinates of any points of intersection with x-axis or the y-axis, and the equation of any asymptotes.
(ii) Shade the region where $(x^2 + y^2 - 9)(x^2 - y^2 - 4) \geq 0$.

2. (a) Find
(i) $\int \frac{1}{(2x+1)^3}\, dx$;
(ii) $\int 4xe^{x^2}\, dx$.
(b) Find $\int \frac{x^2}{x^2+1}\, dx$.
(c) (i) Use the substitution $u = x^2 - 4$ to show that $\int \frac{x}{\sqrt{x^2-4}}\, dx = \sqrt{x^2-4} + c$.
(ii) Hence find the exact value of $\int_{\sqrt{5}}^{\sqrt{8}} \frac{\ln(x^2-4)}{\sqrt{x^2-4}}\, dx$.
(d) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\, dx = \ln 3$.
(ii) Use the substitution $u = \pi - x$ to show that $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}\, dx$.
(iii) Hence find the exact value of $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\, dx$.

3. (a) The line $y = x$ meets a directrix of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ at the point V in the first quadrant. Tangents from V meet the ellipse at $P(x_1, y_1)$ and $Q(x_2, y_2)$. The eccentricity of the ellipse is e.
(i) Show this information on a sketch.
(ii) Given that the chord of contact of tangents from the point (x_0, y_0) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has equation $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$, deduce that the equation of PQ is $\frac{x}{ae} + \frac{y}{ae(1-e^2)} = 1$ and verify that PQ is a focal chord of the ellipse. Show the foci of the ellipse on your sketch.
(iii) Show that z_1 and x_2 are roots of the equation

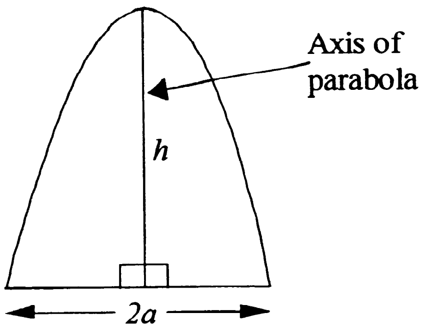
$$(2 - e^2)x^2 - 2ae(1 - e^2)x + a^2(e^2 - e^4 - 1) = 0.$$

(iv) Show that the midpoint M of the chord PQ lies on the line $y = x$.
(v) If the ellipse has foci $S(2, 0)$ and $S'(-2, 0)$ and directrices $x = 8$ and $x = -8$, on a new diagram, sketch the ellipse showing the line $y = x$ and the positions of V, P, Q, S, S' and M. Give the coordinates of V and M and the equation of the ellipse.
(b) $\arg(z - 2) = \arg z$.
(i) Show vectors representing $z, z - 2$ in an Argand diagram.
(ii) If the point P represents z, O is the origin and Q has coordinates $(2, 0)$ in this Argand diagram, what is the nature of $\triangle OPQ$ for non-real z? Deduce that if z is non-real, then P lies on a circle and state its centre and radius.
(iii) On a new diagram, sketch the locus in the Argand diagram of a point representing z satisfying $\arg(z - 2) = 2 \arg z$, for both real and non-real z.

4. (a) (i) Show the roots of $z^5 + 1 = 0$ on a unit circle in an Argand diagram.
(ii) Factor $z^5 + 1$ into irreducible factors with real coefficients.
(iii) Deduce that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}$.
(iv) Write a quadratic equation with integer coefficient which has roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ as surds.
(b) $P(x) = 16x^4 - 32x^3 + 16x^2 + kx - 5$, where k is an integer. $P(x)$ has two rational roots which are opposites of each other, and two non-real roots.
(i) If α is a non-real root of $P(x)$, show that $\Re(\alpha) = 1$ and $|\alpha| > 1$.
(ii) If the rational roots are $\pm\beta$, deduce that $\beta^2 < \frac{5}{16}$.
(iii) Find the rational roots and the value of k.
(iv) Factor $P(x)$ into irreducible factors with integer coefficients.

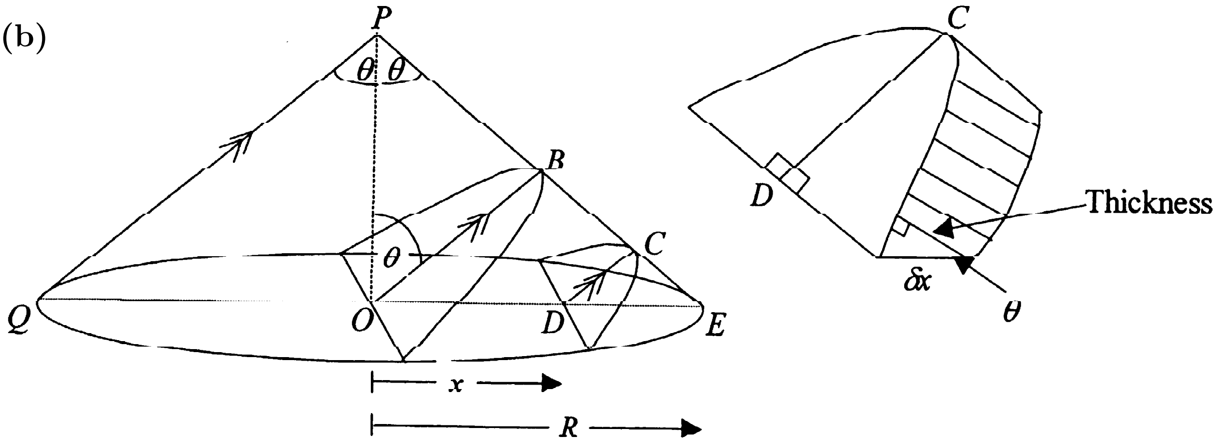
5. (a) $z = 2 - i$. Find real numbers p and q such that $pz + \frac{q}{z} = 1$.
(b) A particle of mass m kg falls from rest in a medium where the resistance to motion is mkv when the particle has velocity v m.s^{-1}.
(i) Draw a diagram showing the forces acting on the particle.
(ii) Show that the equation of motion of the particle is $\ddot{x} = k(V - v)$ where V m.s^{-1} is the terminal velocity of the particle in this medium, and x metres is the distance fallen in t seconds.
(iii) Find in terms of V and k the time T seconds taken for the particle to attain 50% of its terminal velocity, and the distance fallen in this time.
(iv) What percentage of its terminal velocity will the particle have attained in time $2T$ seconds? Sketch a graph of v against t showing this information.
(v) If the particle has reached 87·5% of its terminal velocity in 15 seconds, find the value of k.

6. (a) A parabolic segment has height h and width $2a$.



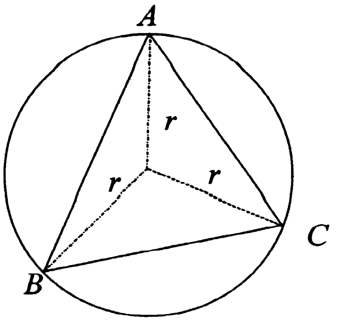
(i) Explain (without calculation) why Simpson's rule with three function values will give the exact area of this segment.
(ii) Use Simpson's rule to show the area is $\frac{4}{3}ah$.

(b)



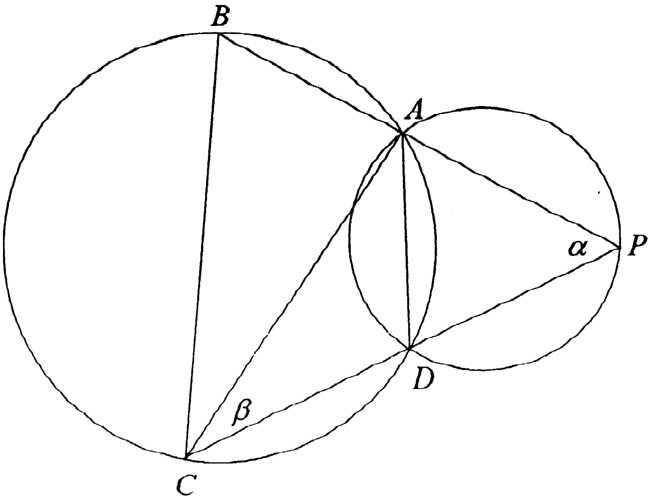
A solid right cone has radius R and semi-vertical angle θ. The cone is sliced by a plane which passes through the centre O of the base of the cone, and makes an angle θ with the axis of the cone. The cross section is the parabolic segment with axis OB shown in the diagram. The volume of the solid cut off the lower right hand corner of the cone by this plane is found by taking slices parallel to the parabolic cross section. A typical slice is the parabolic segment with axis DC shown in the diagram.
(i) If $OD = x$, show that $DC = \frac{R-x}{2\sin\theta}$, and show that the area of the typical slice, a parabolic segment with axis DC, is given by $A = \frac{2(R-x)\sqrt{R^2-x^2}}{3\sin\theta}$.
(ii) Using the fact that this typical slice has thickness $\cos\theta\ \delta x$, deduce that the volume of the solid cut off the cone is given by $V = \frac{2}{3\tan\theta}\int_0^R (R-x)\sqrt{R^2-x^2}\,dx$.
(iii) Hence show $V = \frac{2}{9}\left(\frac{3\pi}{4} - 1\right)R^2 H$ where H is the height of the cone.
(iv) What difference would it have made to the process of finding the volume if the plane slicing the piece off the cone made an angle of $\alpha \neq \theta$ with the axis of the cone? Explain your answer briefly, but do not do any further calculation.

7. **(a)** The circle through the vertices of triangle ABC has centre O and radius r.



(i) Show that $BC = 2r\sin A$.
(ii) Use the fact that Area($\triangle OBC$+Area($\triangle OCA$)+Area($\triangle OAB$) = Area($\triangle ABC$) to show that $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$.

(b)



The two circles intersect at A and D. P is a point on the major arc AD of one circle. The other circle had radius r, and PA produced and PD produced meet the other circle at B and C respectively. $\angle APD = \alpha$ and $\angle ACD = \beta$.
(i) Show that $BC = 2r\sin(\alpha+\beta)$.
(ii) As P moves along the major arc AD on its circle, show that the length of the chord BC in the other circle is a constant.
(iii) If the two circles have equal radii, show that $BC = (2\cos\alpha).AD$.

8. **(a)** A curve is defined by the parametric equations $x = \cos^3\theta, y = \sin^3\theta$ for $0 < \theta < \frac{\pi}{4}$.
(i) Show that the equation of the normal to the curve at the point $P(\cos^3\phi, \sin^3\phi)$ is $x\cos\phi - y\sin\phi = \cos 2\phi$.
(ii) The normal at P meets the x-axis at A and the y-axis at B. Show that $AB = 2\cot 2\phi$.
(b) **(i)** If $x \geq 0$ show that $\frac{2x}{1+x^2} \leq 1$.
(ii) By integrating both sides of this inequality with respect to x between the limits

$x = 0$ and $x = a$, show that $e^a \geq 1 + a^2$ for $a \geq 0$.
(c) A tennis match between two players consist of a number of sets. The match continues until it is won by the player who first wins three sets. Whenever Bill and Boris play tennis against each other, for each set they play there is a probability of $\frac{2}{3}$ that Bill wins the set and a probability of $\frac{1}{3}$ that Boris wins the set. Bill and Boris play a tennis match against each other. Show that the probability that Bill wins the match is $\frac{64}{81}$.
