CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE TRIAL HSC EXAMINATION 4 UNIT MATHEMATICS 1999

Question 1

(a)(i) Show that the x coordinates of the stationary points on the graph $y = \{f(x)\}^2$ are the same as the x coordinates of the stationary points or intercepts on the x axis of the graph $y = f(x)$.

(ii) $f(x) = \sin^2 x - \frac{1}{2}, 0 \leq x \leq \pi$. On the same exaes, and without using further calculus, sketch the graphs of $y = f(x)$ and $y = \{f(x)\}^2$.

(b)(i) Sketch the graph of $y = x^2 + \frac{2}{x}$ showing any intercepts on the coordinate axes, asymptotes, stationary points or points of inflexion.

(ii) The equation $x^2 + \frac{2}{x} - k = 0$ has exactly two different real solutions. Find the value of k and the real solutions of the equation.

(c) The equation $x^3 - x^2 - 3x + 2 = 0$ has roots α, β, γ.

(i) Use the value of $\alpha + \beta + \gamma$ to find the monic polynomial equation with roots $2\alpha + \beta + \gamma, \alpha + 2\beta + \gamma, \alpha + \beta + 2\gamma$.

(ii) Find the monic polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$.

Question 2

(a) Use the substitution $u = x - 1$ to find $\int \frac{x}{\sqrt{x-1}}\ dx$

(b)(i) Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 + \tan^2 x) \tan x\ dx$

(ii) Find the exact value of $\int_0^{\ln 2} e^x \sec^2(e^x)\ dx$

(c) Use the substitution $t = \tan \frac{x}{2}$ to find the exact value of $\int_0^{\frac{\pi}{2}} \frac{1}{7+5\sin x+5\cos x}\ dx$

(d)(i) Use the substitution $u = \frac{1}{x}$ to show that $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\ dx = 0$

(ii) Hence use the substitution $u = \frac{\sqrt{3}}{x}$ to show that $\int_1^3 \frac{\ln x}{3+x^2}\ dx = \frac{\pi\sqrt{3}\ln 3}{36}$

Question 3

(a)(i) Express $z = 2 - 2\sqrt{3}i$ in modulus/argument form.

(ii) Using modulus/argument forms, mark the points representing $\overline{z}, \frac{1}{z}$ on an Argand diagram.

(b)(i) Find the Cartesian equation of the locus represented by $2|z| = 3(z + \overline{z})$.

(ii) Sketch the locus on an Argand diagram.

(c) The hyperbola $xy = c^2$ touches the circle $(x-1)^2 + y^2 = 1$ at the point Q.

(i) Show this information on a sketch.

(ii) Show that if β is a repeated root of the polynomial equation $P(x) = 0$ then β is also a root of $P'(x) = 0$.

(iii) Deduce that the equation $x^2(x-1)^2 + c^4 = x^2$ has a repeated real root $\beta > 0$, and two non-real complex roots.

(iv) Find the values of β, c^2 and the remaining roots of the equation $x^2(x-1)^2 + c^4 = x^2$.

(v) Find the equation of the common tangent to the hyperbola and the circle at Q.

Question 4

(a) $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(i) Verify that the coordinates of P satisfy $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

(ii) Deduce that the chord PQ has equation $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

(iii) Hence or otherwise show that if PQ is a focal chord of the ellipse, then $\cos\frac{\alpha-\beta}{2} = \pm e\cos\frac{\alpha+\beta}{2}$

(b)(i) Show that $\cos(p+q) + \cos(p-q) = 2\cos p\cos q$, and deduce that $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$.

(ii) Show that if PQ is a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where P and Q are $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$, then $PQ = 2a\{1 - e^2\cos^2(\frac{\alpha+\beta}{2})\}$.

(c) $P(a\cos\alpha, b\sin\alpha), Q(a\cos\beta, b\sin\beta)$ and $R(a\cos\theta, b\sin\theta)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. PQ is a focal chord of the ellipse, RT is the diameter through R, and PQ is parallel to RT.

(i) Show this information on a sketch.

(ii) Deduce that T has coordinates $(-a\cos\theta, -b\sin\theta)$ and $RT = 2RO$.

(iii) Show that $RT^2 = 4a^2(1 - e^2 \sin^2 \theta)$.

(iv) Show that $\tan\theta \tan\frac{\alpha+\beta}{2} = -1$.

(v) Show that $RT^2 = 2aPQ$.

Question 5

(a)(i) Show that $\int x \tan^{-1} x \; dx = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + c, c$ constant.

(ii) $I_n = \int_0^1 x^n \tan^{-1} x \; dx, n = 0, 1, 2, \ldots$. Show that $I_0 = \frac{\pi}{4} - \frac{1}{2}\ln 2, I_1 = \frac{\pi}{4} - \frac{1}{2}$ and $I_n = \frac{1}{n+1} \cdot \frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1} \cdot I_{n-2},\ n = 2, 3, 4, \ldots$.

(b)(i) Without using calculus, sketch the graphs $y = \tan^{-1} x, y = x^2 \tan^{-1} x$ on the same axes showing the coordinates of the points of intersection.

(ii) The region bounded by these curves is rotated through one revolution about the y axis. Use the method of cylindrical shells to show the volume V of the solid is given by $V = 4\pi \int_0^1 (x - x^3) \tan^{-1} x \; dx$ and evaluate V.

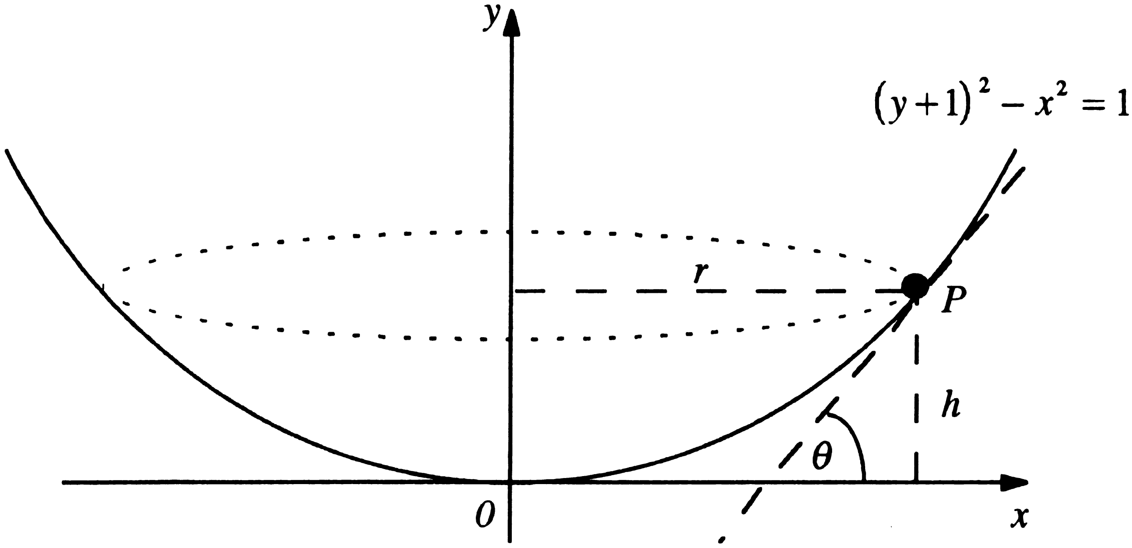
Question 6

(a)(i) Show that the roots of $z^6 - z^3 + 1 = 0$ are amongst the roots of $z^9 + 1 = 0$.

(ii) Show that $z^6 - z^3 + 1 = (z^2 - 2z\cos\frac{\pi}{9} + 1)(z^2 - 2z\cos\frac{5\pi}{9} + 1)(z^2 - 2z\cos\frac{7\pi}{9} + 1)$

(iii) Show that $\cos\frac{\pi}{9}\cos\frac{5\pi}{9} + \cos\frac{5\pi}{9}\cos\frac{7\pi}{9} + \cos\frac{7\pi}{9}\cos\frac{\pi}{9} = -\frac{3}{4}$

(b)



A smooth bowl is formed by rotation the hyperbola $(y+1)^2 - x^2 = 1$ around the y axis. A particle P of mass m kg travels around the inside of the bowl with constant angular velocity ω radians per second in a horizontal circle of radius r metres at a height h metres above the bottom of the bowl. The acceleration due to gravity is g m.s^{-2}

(i) Show that if the tangent to the hyperbola $(y+1)^2 - x^2 = 1$ at the point (x_1, y_1) makes an angle θ with the positive x axis, then $\tan\theta = \frac{x_1}{1+y_1}$.

(ii) Draw a diagram showing the forces on P.

(iii) Show that $\omega^2 = \frac{g}{1+h}$.

(iv) Show that the force N Newtons exerted by the particle P on the bowl is given by $N = mg\sqrt{2 - \frac{1}{(1+h)^2}}$.

(v) If the linear speed of the particle is $\sqrt{\frac{3g}{2}}$ m.s^{-1}, find h and the force exerted by the particle on the bowl.

Question 7

(a)(i) Show that for all values of A and B, $\sin(A+B) - \sin(A-B) = 2\cos A\cos B$

(ii) Use the method of Mathematical Induction to show that for all positive integers n, $\cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$.

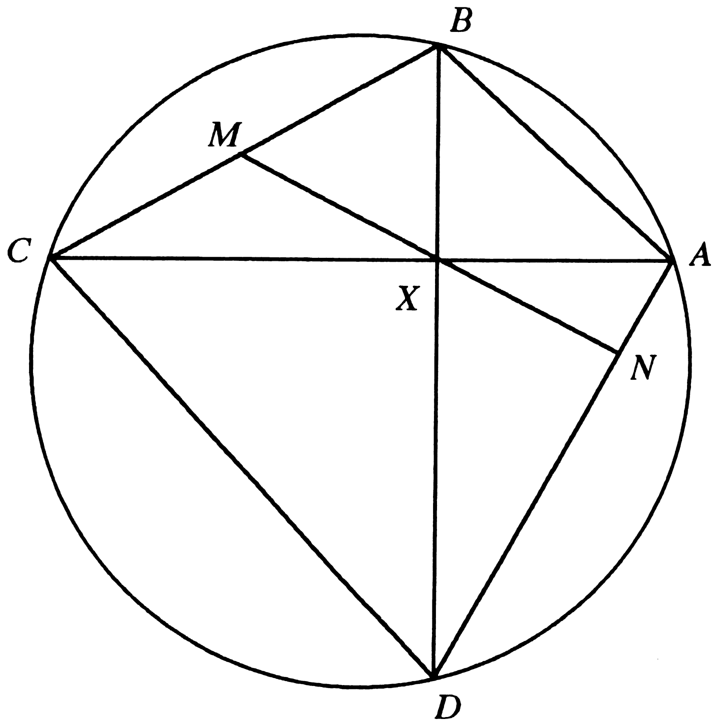
(iii) Hence show that $\cos 2x + \cos 4x + \cos 6x + \cdots + \cos 16x = 8\cos 9x\cos 4x\cos 2x\cos x$

(b)(i) If $x > 0, y > 0$ show that $x + y \geq 2\sqrt{xy}$.

(ii) Hence show that if $x > 0, y > 0, z > 0$ then $(x + y)(y + z)(z + x) \geq 8xyz$.

(iii) If a, b, c are the sides of a triangle with semi perimeter $S = \frac{1}{2}(a + b + c)$ then Heron's formula states that the area Δ of the triangle is given by $\Delta = \sqrt{S(S - a)(S - b)(S - c)}$. By choosing suitable values for x, y, z show that $\Delta^2 \leq \frac{(a+b+c)abc}{16}$.

Question 8



$ABCD$ is a cyclic quadrilateral. The diagonals AC and BD intersect at right angles at X. M is the midpoint of BC. MX produced meets AD at N.

(i) Copy the diagram showing the above information.

(ii) Show that $M\hat{B}X = M\hat{X}B$

(iii) Show that MN is perpendicular to AD.

(b)(i) Find the probability that 6 throws of a fair die result in exactly 3 even scores.

(ii) Find the probability that 6 throws of a fair die result in exactly 3 even scores, all of which are different.

(iii) Find the probability that exactly 6 throws of a fair die are needed in order to obtain 3 even scores.

(iv) Find the probability that at least 6 throws of a fair die are needed in order to obtain 3 even scores.