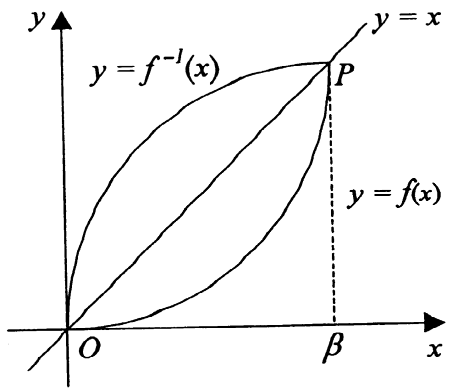
CSSA of NSW

2000 Trial HSC 4 Unit Mathematics

1. **(a)** The diagram shows the graphs of $y = f(x)$, and its inverse $y = f^{-1}(x)$. The graphs intersect in $(0, 0)$ and in the point P with x coordinate β.



Use the substitution $u = f^{-1}(x)$ to show that $\int_0^\beta f^{-1}(x)\ dx = \int_0^\beta f'(u)\ du$ and hence show that the area bounded by $y = f(x)$ and $y = f^{-1}(x)$ is given by $A = \int_0^\beta \{xf'(x) - f(x)\}\ dx$.

(b) $y = x\sin^{-1} x$

(i) Show that $\frac{dy}{dx} = \sin^{-1} x + \tan(\sin^{-1} x)$.

(ii) By considering the graph of $y = \tan\theta$, deduce that the graph of $y = x\sin^{-1} x$ has exactly one stationary point. Show this stationary point is a minimum turning point at $(0, 0)$.

(iii) Sketch the graph of $y = \sin^{-1} x$. Show the nature of the curve near the endpoints of its domain.

(iv) If $f(x) = x\sin^{-1} x, x \geq 0$, show on a new diagram the graph of $y = f(x)$, its inverse $y = f^{-1}(x)$, and the line $y = x$. Give the coordinates of any points of intersection and of the endpoints of the curves.

(v) Use the result in **(a)** to show that the area bounded by the curves $y = f(x)$ and $y = f^{-1}(x)$ between their point of intersection in given by $\int_0^{\sin 1} \frac{x^2}{\sqrt{1-x^2}}\ dx$. Use the substitution $x = \sin\theta$ to evaluate this area.

2. **(a)** **(i)** Find the two square roots of $2i$.

(ii) Solve $x^2 + 2x + (1 - \frac{1}{2}i) = 0$.

(b) Find:

(i) $\int (1 + \tan^2 x)e^{\tan x}\ dx$.

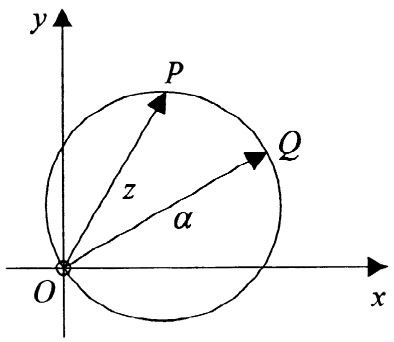
(ii) $\int te^{-t}\ dt$.

(iii) $\int \cos^3 x\ dx$.

(c) Evaluate in simplest exact form $\int_e^{e^2} \frac{1}{x\ln x}\ dx$.

(d) Use the substitution $t = \tan \frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\, dx$, giving your answer in simplest exact form.
(e) If $I = \int_e^{\ln 2} \frac{e^x}{e^x+e^{-x}}\, dx$ and $J = \int_e^{\ln 2} \frac{e^{-x}}{e^x+e^{-x}}\, dx$, find the exact values of $I + J$ and $I - J$ and hence find the exact values of I and J.

3. **(a)** $z_1 = 1 + 2i$ and $z_2 = 3 - i$. Find the value of $z_1^2 \div \overline{z}_2$.
(b) $z = \sqrt{3} + i$
(i) Write z in modulus/argument form.
(ii) What can you say about integers n such that $z^n + (\overline{z})^n$ is rational?
(iii) Find the smallest positive integer n such that $z^n + (\overline{z})^n$ is a negative rational number, and for this value of n, state the value of $z^n + (\overline{z})^n$.
(c) $\alpha = p + iq$ where p and q are real.
(i) If z satisfies $\Re(\alpha z) = 1$, show that the locus of the point P representing z in the Argand diagram is the line $px - qy = 1$.
(ii)

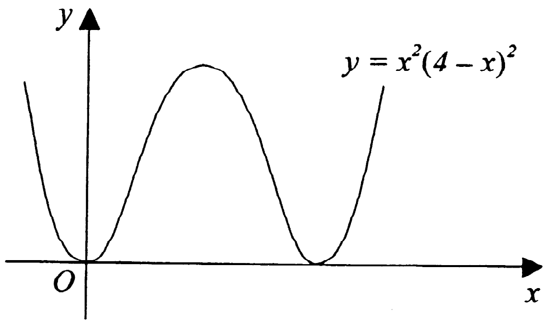


The vector $\overrightarrow{OQ}$ represents α in the Argand diagram. If $z \neq 0$ is represented by the vector $\overrightarrow{OP}$ where P lies on the circle with diameter OQ, copy the diagram and show the vector representing $z - \alpha$. Show that for such a complex number z, $\frac{z-\alpha}{z}$ is imaginary and hence $\Re(\alpha \frac{1}{z}) = 1$.
(iii) Deduce that if z is a non-zero complex number such that the point P repesenting z in the Argand diagram lies on the circle with diameter OQ, where Q has coordinates (p, q), then the point representing $\frac{1}{z}$ in the same Argand diagram lies on the line $px - qy = 1$.
(iv) $z \neq 0$ satisfies the condition $|z - (1 + i)| = \sqrt{2}$. Sketch the locus of the points representing z and $\frac{1}{z}$ in the same Argand diagram, and label each locus with its equation. Considering only values between $-\pi$ and π, what are the possible values of $\arg(z)$?

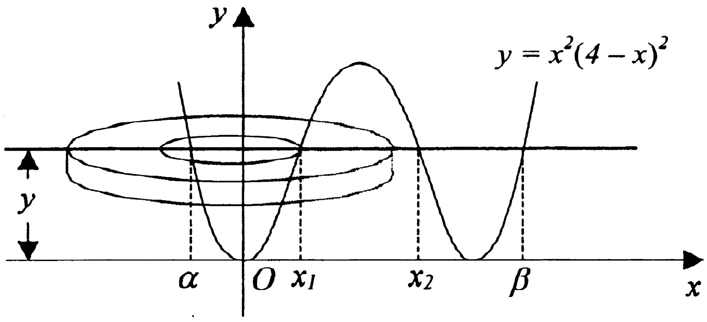
4. Hyperbola H has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity e, while ellipse E has equation $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$.
(i) Show that E has eccentricity $\frac{1}{e}$.
(ii) Show that E passes through one focus of H, and H passes through one focus of E.

(iii) Sketch H and E on the same diagram, showing the foci S, S', of H and T, T' of E, and the directrices of H and E. Give the coordinates of the foci and the equations of the directrices in terms of a and e.
(iv) If H and E intersect at P in the first quadrant, show that the acute angle α between the tangents to the curves at P satisfies $\tan\alpha = \sqrt{2}(e + \frac{1}{e})$.
(v) What is the smallest possible acute angle between the tangents to the curves H and E at their point of intersection P?
(vi) Find the acute angle between the tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at their points of intersection. Give your answer to the nearest degree.

5. **(a)** Show that the stationary point of $y = \{f(x)\}^2$ are exactly those point on the curve that have x coordinates which are zeros of either $f(x)$ or $f'(x)$.
(b)



Use the graph of $y = x(4 - x)$ to justify the features shown on the graph above. Copy the graph of $y = x^2(4 - x)^2$ and mark on the coordinate axes the values of x and y at the stationary points.
(c)



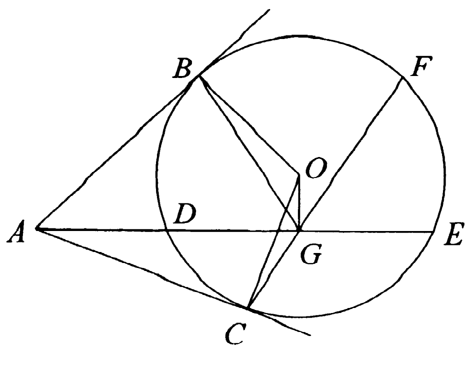
The shaded region is rotated through one revolution about the y axis. The volume of the solid formed is found by taking slicing perpendicular to the y axis. The typical slice shown in the diagram is at a height y above the x axis.
(i) Deduce that α, x_1, x_2 and β, as shown in the diagram, are roots of $x^4 - 8x^3 + 16x^2 - y = 0$.
(ii) Use the symmetry in the graph to explain why $\frac{x_1+x_2}{2} = 2$ and $\frac{\alpha+\beta}{2} = 2$. Hence, by considering the coefficients of the equation in **(i)**, show that $\alpha\beta = -x_1x_2$ and

deduce that $x_1x_2 = \sqrt{y}$ and $x_2 - x_1 = 2\sqrt{4-\sqrt{y}}$.
(iii) Show that the volume of the solid of revolution is given by $V= 8\pi\int_0^{16}\sqrt{4-\sqrt{y}}\,dy$. Use the substitution $y = (4-u)^2$ to evaluate this integral and find the exact volume.

6. A toy of mass m kg has a parachute device attached. It is released from rest at the top of a vertical cliff 40 m high. During its fall, the forces acting are gravity and, owing to the parachute, a resistance force of magnitude $\frac{1}{10}mv^2$ when the speed of the toy is v ms^{-1}. After $2\ln 2$ seconds, the parachute disintegrates, and then the only force acting on the toy is gravity. The acceleration due to gravity is taking as $g = 10$ ms^{-2}. At time t seconds, the toy has fallen a distance x metres from the top of the cliff, and its speed is v ms^{-1}.
(i) Show that while the parachute is operating, $10\ddot{x} = 100 - v^2$. Hence show that $v = \left(\frac{e^{2t}-1}{e^{2t}+1}\right)$ and $x = -5\ln\left[1-\left(\frac{v}{10}\right)^2\right]$.
(ii) Find the exact speed of the toy and the exact distance fallen just before the parachute disintegrates.
(iii) After the parachute disintegrates, find an expression for $\ddot{x}$ and use integration to find the speed of the toy just before it reaches the base of the cliff. Give your answer correct to 2 significant figures.

7. (a) The equation $x^3 + px - 1 = 0$ has three real, non-zero roots α, β, γ.
(i) Find the value of $\alpha^2+\beta^2+\gamma^2$ and $\alpha^4+\beta^4+\gamma^4$ in terms of p, and hence show that p must be strictly negative.
(ii) Find the monic equation, with coefficients in terms of p, whose roots are $\frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta}$.
(b) (i) If $I_n = \int_0^1 (x^2-1)^n\,dx, n = 0,1,2,3,\ldots$, show that $I_n = \frac{-2n}{2n+1}I_{n-1}, n = 1,2,3,\ldots$
(ii) Hence use the method of Mathematical Induction to show that $I_n = \frac{(-1)^n 2^{2n}(n!)^2}{(2n+1)!}$ for all positive integers n.

8. (a)



In the diagram, AB and AC are tangents from A to the circle with centre O, meeting the circle at B and C. ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the cicle at F.

(i) Copy the diagram.
(ii) Show that $ABOC$ and $AOGC$ are cyclic quadrilaterals.
(iii) Show that $BF||ADE$.
(b) (i) If $y = x^k + (c-x)^k$, where $c > 0, k > 0, k \neq 1$, show that y has a single stationary value between $x = 0$ and $x = c$, and show that this stationary value is a maximum if $k < 1$ and a minimum if $k > 1$.
(ii) Hence show that if $a > 0, b > 0, a \neq b$, then $\frac{a^k+b^k}{2} < \left(\frac{a+b}{2}\right)^k$, if $0 < k < 1$, and $\frac{a^k+b^k}{2} > \left(\frac{a+b}{2}\right)^k$ if $k > 1$.