

Name : _____

Class : 12 MTZ ____

Pymble Ladies' College
Mathematics Department

KW

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2001 AP4

YEAR 12 TRIAL HSC

MATHEMATICS EXTENSION II

[4 UNIT]

*Time allowed - 3 hours
(plus 5 minutes reading time)*

DIRECTIONS TO CANDIDATES:

- Attempt ALL questions.
- All questions are of equal value.
- Standard Integrals are provided.
- Approved calculators may be used.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Each question attempted is to be returned on a new page clearly marked Question 1, Question 2, etc on the top of the page.
- Each page must show your class and your name.

Students are advised that this is a school based Trial Examination *only* and cannot in any way guarantee the complete content nor format of the Higher School Certificate Examination.

QUESTION 1. (15 marks) Use a SEPARATE writing booklet.

Marks

(a) Evaluate $\int_0^3 \frac{x \, dx}{\sqrt{16+x^2}}$.

3

(b) Find $\int \frac{dx}{x^2+6x+13}$.

2

(c) Find $\int xe^{-x} \, dx$.

2

(d) Find $\int \cos^3 \theta \, d\theta$.

3

(e) (i) Find constants A , B and C such that

3

$$\frac{x^2-4x-1}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}.$$

(ii) Hence find $\int \frac{x^2-4x-1}{(1+2x)(1+x^2)} \, dx$.

2

QUESTION 2. (15 marks) Use a SEPARATE writing booklet.

Marks

(a) Given that $z = 1 + i$ and $w = -3$, find, in the form $x + iy$:

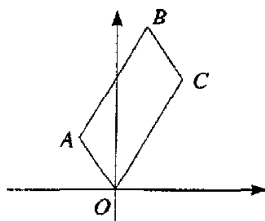
(i) wz^2 , 1

(ii) $\frac{z}{z+w}$. 2

(b) Using de Moivre's theorem, simplify $(-1 - i\sqrt{3})^{-10}$, expressing the answer in the form $x + iy$. 3

(c) Find the values of real numbers a and b such that $(a + ib)^2 = 5 - 12i$. 2

(d) 3



In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$.

The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

If $\angle AOC = 60^\circ$, what complex number does C represent?

(e) z_1 and z_2 are complex numbers.

(i) Show that $|z_1| |z_2| = |z_1 z_2|$. 1

(ii) By taking $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$, express 533 (the product of 13 and 41) as a sum of squares of two positive integers. 1

(iii) By taking other values for z_1 and z_2 , express 533 as a sum of squares of two other positive integers. 2

QUESTION 3. (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) On separate number planes, draw graphs of the following functions, showing essential features.
- (i) $y = \frac{x+1}{x-1}$ 2
- (ii) $y = \sqrt{\frac{x+1}{x-1}}$ 2
- (iii) $y = \ln\left(\frac{x+1}{x-1}\right)$ 2
- (b) z is a variable complex number which is represented by the point P . Find the locus of P if $|z - 2i| = \text{Im}(z)$ 2
- (c) The fixed complex number α is such that $0 < \arg \alpha < \frac{\pi}{2}$. In an Argand diagram α is represented by the point A while $i\alpha$ is represented by the point B . z is a variable complex number which is represented by the point P .
- (i) Draw a diagram showing A , B and the locus of P if $|z - \alpha| = |z - i\alpha|$. 1
- (ii) Draw a diagram showing A , B and the locus of P if $\arg(z - \alpha) = \arg(i\alpha)$. 1
- (iii) Find in terms of α the complex number represented by the point of intersection of the two loci in (i) and (ii). 1
- (d) Consider the function $y = \sin^{-1}(e^{-x})$.
- (i) Find the domain and range of the function. 2
- (ii) Sketch the graph of the function showing clearly the coordinates of any endpoints and the equations of any asymptotes. 2

QUESTION 4. (15 marks) Use a SEPARATE writing booklet.

Marks

Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

- (a) (i) Find the eccentricity of the ellipse. **1**
(ii) Find the coordinates of the foci and the equations of the directrices of the ellipse. **2**
(iii) Sketch the graph of the ellipse showing clearly all of the above features and the intercepts on the coordinate axes. **2**
- (b) (i) Use differentiation to derive the equations of the tangent and the normal to the ellipse at the point $P(2,3)$. **3**
(ii) The tangent and normal to the ellipse at P cut the y axis at A and B respectively. Find the coordinates of A and B . **1**
- (c) (i) Show that AB subtends a right angle at the focus S of the ellipse. **2**
(ii) Show that the points A, P, S and B are concyclic. **1**
(iii) Find the centre and radius of the circle which passes through the points A, P, S and B . **3**

QUESTION 5. (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) (i) Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. **2**
(ii) Hence or otherwise find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. **2**
(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. You do not need to give the coordinates of turning points or inflections. **1**
- (b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by $\theta = (2n+1)\frac{\pi}{5}$, $n=0, \pm 1, \pm 2, \dots$.
Hence solve the equation $\cos 5\theta = -1$ for $0 \leq \theta \leq 2\pi$. **2**
(ii) Use De Moivre's Theorem to show that $\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$. **3**
(iii) Find the exact trigonometric roots of the equation $16x^5 - 20x^3 + 5x + 1 = 0$. **2**
(iv) Hence find the exact values of $\cos \frac{\pi}{5} + \cos \frac{3\pi}{5}$ and $\cos \frac{\pi}{5} \cdot \cos \frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1$ into irreducible factors over the rational numbers. **3**

QUESTION 6. (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) A lifebelt mould is made by rotating the circle $x^2 + y^2 = 64$ through one complete revolution about the line $x = 28$, where all the measurements are in centimetres.
- (i) Use the method of slicing to show that the volume $V \text{ cm}^3$ of the lifebelt is given by 5
- $$V = 112 \pi \int_{-8}^8 \sqrt{64 - y^2} \, dy.$$
- (ii) Find the exact volume of the lifebelt. 2
- (b) (i) Show that $\frac{t^n}{1+t^2} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$. 1
- (ii) Let $I_n = \int \frac{t^n}{1+t^2} dt$. 1
- Show that $I_n = \frac{t^{n-1}}{n-1} - I_{n-2}$, $n \geq 2$.
- (iii) Show that $\int_0^1 \frac{t^6}{1+t^2} dt = \frac{13}{15} - \frac{\pi}{4}$. 3
- (c) In a series of five games played by two equally matched teams, team A and team B, the team that wins three games first is the champion.
- (i) If team B wins the first two games, what is the probability that team A is the champion? 1
- (ii) If team A has won the first game, what is the probability that team A is the champion? 2

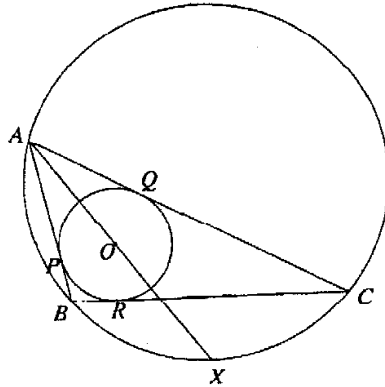
QUESTION 7. (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) In the diagram below, ABC is a triangle.

The incircle tangent to all three sides has centre O , and touches the sides AB , AC and BC at P , Q and R respectively.

The circumcircle through A , B and C meets the line AO produced at X .



- (i) Show that $\angle CBX = \angle CAX$. 1
- (ii) Use congruence to prove that $\angle OBA = \angle OBC$. 2
- (iii) Prove that $\triangle XBO$ is an isosceles triangle. 3
- (iv) Prove that $BX = XC$. 1
- (b) (i) α . Differentiate $y = \log_e(1+x)$, and hence draw $y = x$ and $y = \log_e(1+x)$ on one set of axis. 1
- β . Using this graph, explain why $\log_e(1+x) < x$, for all $x > 0$. 1
- (ii) α . Differentiate $y = \frac{x}{1+x}$, and hence draw $y = \frac{x}{1+x}$ and $y = \log_e(1+x)$ on one set of axis. 1
- β . Using this graph, explain why $\frac{x}{1+x} < \log_e(1+x)$, for all $x > 0$. 1
- (iii) Use the inequalities of parts (i) and (ii) to show that 4

$$\frac{\pi}{8} - \frac{1}{4} \log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2} dx < \frac{1}{2} \log_e 2.$$

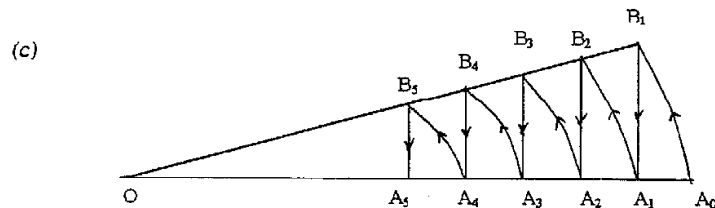
QUESTION 8. (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) At a dinner party there are twelve people, consisting of the six State Premiers and their partners. Each couple was representing one of the six States: New South Wales, Victoria, Western Australia, South Australia, Tasmania and Queensland.
- (i) The dinner took place at a circular table. Find how many seating arrangements are possible if:
- α. there are no restrictions, 1
- β. the males and females are in alternate positions. 1
- (ii) A committee of six is to be formed from the Premiers and their partners, where not more than one State can have two representatives. How many such committees are possible? 2

- (b) It is given that if a, b, c are any three positive real numbers, then $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.
 If $a > 0$, $b > 0$ and $c > 0$ are real numbers such that $a + b + c = 1$, use the given result to show that

- (i) $\frac{1}{abc} \geq 27$ 1
- (ii) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$ 2
- (iii) $(1-a)(1-b)(1-c) \geq 8abc$ 2



An ant walks along the circular arc from A_0 to B_1 , then down the straight line to A_1 , along the circular arc to B_2 , then down to A_2 , and so on, until it reaches O .

The length of OA_0 is 1, while angle A_0OB_1 is x radians, $0 < x \leq \frac{\pi}{2}$.

- (i) Show that the total distance the ant walks by the time it reaches O is given
 by $y = \frac{x + \sin x}{1 - \cos x}$ 2
- (ii) Find the derivative of y with respect to x and explain why the derivative of y is always negative for all $0 < x \leq \frac{\pi}{2}$ 2
- (iii) Hence find the shortest possible distance the ant needs to walk from A_0 to O . 2

End of Paper