CRANBROOK SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2001

MATHEMATICS

4 UNIT (Additional)

Time allowed – Three hours

DIRECTIONS TO CANDIDATES

* Attempt all questions.
* ALL questions are of equal value.
* All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
* Standard integrals are printed on the back page.
* Board-approved calculators may be used.
* You may ask for extra Writing Booklets if you need them.

* Submit your work in four 8 page booklets :

(i) QUESTIONS 1 & 2

(ii) QUESTIONS 3 & 4

(iii) QUESTIONS 5 & 6

(iv) QUESTIONS 7 & 8

1. (8 page booklet)

(a) Find (i) $\int \cot x \ \operatorname{cosec}^2 x \ dx$ (ii) $\int \frac{\sec^2 x}{3-\tan x} dx$ **[4 marks]**

(b) Prove that $\int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{(x-5)(7-x)}} = \frac{\pi}{3}$, by using the substitution $u = x-6$. **[3 marks]**

(c) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **[2 marks]**

(ii) Hence or otherwise evaluate $\int_0^{\pi/2} \frac{\cos^5 x}{\cos^5 x + \sin^5 x} dx$. **[2 marks]**

(d) Evaluate $\int_0^{\pi/4} \frac{dx}{2\sin 2x + \cos x}$ **[4 marks]**

2.

(a) Evaluate $\int_{-\pi/4}^{\pi/4} \frac{x^3}{\cos x} dx$ **[2 marks]**

(b) Find $\int \sin^3 2x \ \cos^2 2x \ dx$ **[3 marks]**

(c) Find $\int \frac{4x-3}{\sqrt{6+2x-3x^2}} \ dx$ **[4 marks]**

(d) If $I_n = \int_0^{\pi/2} \cos^n x \sin^2 x \, dx$ for $n \geq 0$, show that $I_n = \frac{n-1}{n+2} I_{n-2}$ for $n \geq 2$. **[4 marks]**

Hence or otherwise evaluate $\int_0^{\pi/2} \cos^4 x \sin^2 x \, dx$. **[2 marks]**

3. (new 8 page booklet please)

(a) (i) Given $z_1 = 1-i$ and $z_2 = -1+\sqrt{3}i$ evaluate $|z_1 z_2|$ and $\arg(z_1 z_2)$

(ii) Find $z_1 z_2$ in cartesian form, and hence show that $\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ **[6 marks]**

(b) If z is a complex number for which $|z| = 1$ show that

(i) $1 \leq |z+2| \leq 3$ and (ii) $-\frac{\pi}{6} \leq \arg(z+2) \leq \frac{\pi}{6}$ **[4 marks]**

(c) (i) Given that $z + \frac{1}{z} = k$, a real number, show that z lies either on the real axis or on the unit circle, centre the origin.

(ii) If z lies on the real axis, show that $|k| \geq 2$; if z lies on the unit circle, show that $|k| \leq 2$. **[5 marks]**

4.

(a) Find integers a and b such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + ax + b$. **[3 marks]**

(b) The equation $z^2 + (1+i)z + k = 0$ has a root $1-2i$. Find the other root, and the value of k. **[3 marks]**

(c) Let α, β, γ be the roots (none of which is zero) of $x^3 + 3px + q = 0$.

(i) Obtain the monic equation whose roots are $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}$.

(ii) Deduce that $\gamma = \alpha\beta$ if and only if $(3p-q)^2 + q = 0$ **[9 marks]**

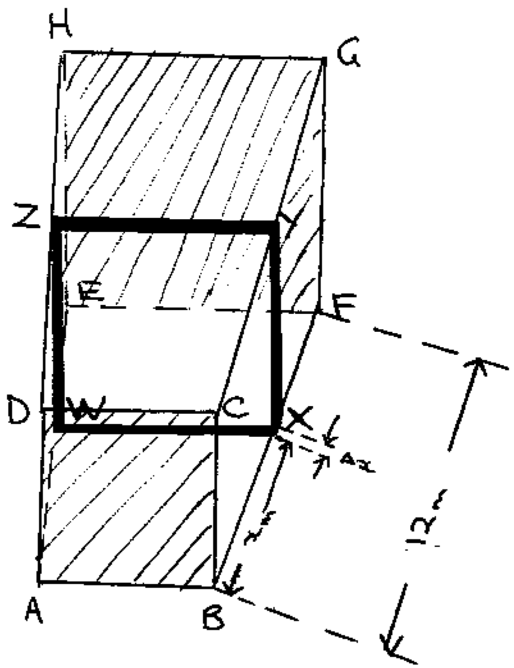
5. (new 8 page booklet please)

(a) The region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 3x$ is rotated about the x-axis. By including appropriate diagrams in each case, find the volume of the solid of revolution by using :

(i) circular discs **[5 marks]**

(ii) cylindrical shells. **[5 marks]**

(b) In the solid shown ABCD and EFGH are squares of side 6 m and 10 m respectively. BCGF is a ~~parallelogram~~ trapezium of height 12 m. Cross-sections parallel to the ends are squares. Show that at a distance x m from the base AB the area of the cross-section is $\left(6+\frac{x}{3}\right)^2$. Hence, by taking slices of thickness Δx find the total volume of the solid. **[5 marks]**



6.

(a) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ are points on the rectangular hyperbola $xy = 9$. The equation of chord PQ is $x + pqy = 3(p+q)$.

(i) Find the co-ordinates of N, the midpoint of PQ.

(ii) If chord PQ is a tangent to the parabola $y^2 = 3x$ prove that the locus of N is $3x = -8y^2$.

[5 marks]

(b) A cylinder of constant volume V has its radius increasing at 5% per minute. At what % rate is the height diminishing?

[4 marks]

(c) A cyclist and a jogger journey along two roads OA and OB, which are inclined at 60^0 to one another. The cyclist starts at a point P, 10 km from O along OA and cycles towards O. At the same instant the jogger starts from O and runs away from O along OB. If the cyclist travels at 8 km/h and the jogger runs at 5 km/h find the rate at which the distance between the two is changing after 90 minutes (in km/h), correct to 2 decimal places.

[6 marks]

7. (new 8 page booklet please)

(a) Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ intersect at right angles.

[5 marks]

(b) You are given that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad (a^2 > b^2)$ at the point $P(x_1, y_1)$ is $a^2 y_1 x - b^2 x_1 y = \left(a^2 - b^2\right) x_1 y_1$.

(i) This normal meets the major axis of the ellipse at G. S is a focus of the ellipse. Show that $GS = e \times PS$, where e is the eccentricity of the ellipse.

[5 marks]

(ii) The normal at the point $P(5\cos\theta, 3\sin\theta)$ on $\frac{x^2}{25} + \frac{y^2}{9} = 1$ cuts the major and minor axes of the ellipse at G and H respectively. Show that as P moves on the ellipse, the mid-point of GH describes another ellipse with the same eccentricity as the first.

[5 marks]

8.

(a) In a certain cricket club there are 15 players available for selection, including 2 Smith brothers, 3 Brown brothers and 10 others. In how many ways may an eleven be selected for a game, if no more than 1 Smith and 2 Browns may be chosen?

[3 marks]

(b) Given that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ are all acute

(i) show that $\quad \sin\left[\sin^{-1} x - \cos^{-1} x\right] = 2x^2 - 1$

(ii) solve the equation $\quad \sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$

[5 marks]

(c) The equation of a curve is $x^2y^2 - x^2 + y^2 = 0$.

(i) Show that the numerical value of y is always less than 1.

(ii) Find the equations of the asymptotes.

(iii) Show that $\frac{dy}{dx} = \frac{y^3}{x^3}$

(iv) Sketch the curve.

[7 marks]

STANDARD INTEGRALS

$\int x^n dx = \frac{1}{n+1} x^{n+1} \quad (n \neq -1; x \neq 0 \text{ if } n < 0)$

$\int \frac{1}{x} dx = \log_e x \quad (x > 0)$

$\int e^{ax} dx = \frac{1}{a} e^{ax} \quad (a \neq 0)$

$\int \cos ax\, dx = \frac{1}{a} \sin ax \quad (a \neq 0)$

$\int \sin ax\, dx = -\frac{1}{a} \cos ax \quad (a \neq 0)$

$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax \quad (a \neq 0)$

$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax \quad (a \neq 0)$

$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad (a \neq 0)$

$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \quad (a > 0, -a < x < a)$

$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log_e \left\{x + \sqrt{x^2 - a^2}\right\} \quad (|x| > |a|)$

$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log_e \left\{x + \sqrt{x^2 + a^2}\right\}$